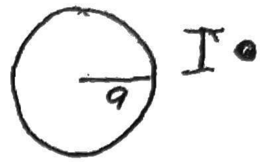
Phys 332 HW-1 Solutions




① $U = \iiint B^2/2\mu_0 \, d\tau$ I need $B_{in}$ and $B_{out}$

$$\oint \vec{B}\cdot d\vec{s} = \mu_0 I_{en} \qquad B(2\pi\rho) = \mu_0 I \frac{\rho^2}{a^2} \qquad \vec{B}_{in} = \frac{\mu_0 I \rho^2}{2\pi a^2}\hat{\phi}$$

$$B(2\pi\rho) = \mu_0 I \qquad \vec{B}_{out} = \frac{\mu_0 I}{2\pi\rho}\hat{\phi}$$

$$U_{in} = \int_0^{l}\int_0^{2\pi}\int_0^{a} \frac{\mu_0^2 I^2 \rho^2}{4\pi^2 a^4} \frac{1}{2\mu_0} \rho\, d\rho\, d\phi\, dz = \frac{\mu_0 I^2}{8\pi^2 a^4} \frac{1}{4} a^4 (2\pi)(l)$$

or $U_{IN} = \dfrac{\mu_0 I^2 l}{16\pi}$

$$U_{out} = \int_0^{l}\int_0^{2\pi}\int_a^{R} \frac{\mu_0^2 I^2}{4\pi^2 \rho^2} \frac{1}{2\mu_0} \rho\, d\rho\, d\phi\, dz$$

$$= \frac{\mu_0 I^2}{8\pi^2} \ln\frac{R}{a} (2\pi)(l)$$

or $U_{out} = \dfrac{\mu_0 I^2 l}{4\pi} \ln\dfrac{R}{a}$

$$\boxed{\text{or } U_{total} = \frac{\mu_0 I^2 l}{16\pi}\left(1 + 4\ln\frac{R}{a}\right)}$$

② $U = \frac{1}{2}\int_{\text{all space}} \vec{J}\cdot\vec{A}\, d\tau$ $\vec{J}$ is really a surface current here specifically $\vec{K} = nI\,\hat{\phi}$

now $\iint \vec{B}\cdot d\vec{a} = \oint \vec{A}\cdot d\vec{s}$ since $\vec{K}$ is only on the surface the integral for U is 2D — so I need $\vec{A}$ on the surface

$\downarrow$

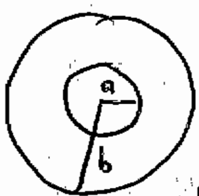
by analogy w/ Ampere: $B = \mu_0 n I$ (see class notes)

$$B(\pi a^2) = A(2\pi a) \qquad \mu_0 n I(\pi a^2) = A(2\pi a)$$

so $\vec{A} = \mu_0 n I a/2 \; \hat{\phi}$ $\qquad U = \frac{1}{2}\int_0^{2\pi}\int_0^{l} n I \hat{\phi} \cdot \frac{\mu_0 n I a}{2}\hat{\phi}\, dz\, a\, d\phi$

$= \frac{1}{2}\mu_0 n^2 I^2 a^2 \frac{1}{2}(2\pi) l$ $\qquad U_{total} = \frac{\mu_0 n^2 I^2 a^2 \pi l}{2}$

or $\underline{U_{total} = \frac{1}{2}\mu_0 n^2 I^2 A l}$ where $A = \pi a^2$ agrees w/ text

(3) 

$U = \frac{1}{2} L I^2$ $\qquad U = \int_{\text{all space}} B^2/2\mu \, d\tau$ $\qquad$ B is only nonzero between a and b

wangsness 17-24 $\qquad \oint \vec{B}\cdot d\vec{s} = \mu_0 I_{en}$ $\qquad B(2\pi\rho) = \mu_0 I$ so $\vec{B} = \frac{\mu_0 I}{2\pi\rho}\hat{\phi}$

$$U = \frac{1}{2\mu_0}\int_0^{l}\int_0^{2\pi}\int_a^{b} \frac{\mu_0^2 I^2}{4\pi^2\rho^2}\rho\, d\rho\, d\phi\, dz = \frac{\mu_0 I^2}{8\pi^2}\ln\frac{b}{a}(2\pi)(l)$$

$$U = \frac{\mu_0 I^2 l}{4\pi}\ln\frac{b}{a} = \frac{1}{2} L I^2 \quad \text{so} \quad \boxed{L = \frac{\mu_0 l}{2\pi}\ln\frac{b}{a}}$$

<u>Magnetic pressure</u>: $\frac{F_m}{A} = f_m = U_m = \frac{B^2}{2\mu_0} = \left(\frac{\mu_0 I}{2\pi\rho}\right)^2 \frac{1}{2\mu_0}\Big|_{\rho=a}$

$\boxed{f_m = \frac{\mu_0 I^2}{8\pi^2 a^2}}$

the forces from the currents at $\rho = b$ all cancel. The forces from the other currents at $\rho = a$ all add and they are all attractive (parallel currents!) – so the pressure tries to collapse the shell. $f_m = 1\times10^5 \frac{N}{m^2} = \frac{\mu_0 I^2}{8\pi^2 (.01m)^2}$

$\boxed{\rightarrow I = 25000 A}$

Wangsness 19-1

4 so this current is clearly not in a plane $\vec{r} = a\hat{\rho} + b\sin n\phi\,\hat{z}$

the wire "oscillates" up and down along z and completes n cycles by the time it wraps around the cylinder

$\vec{m} = \frac{1}{2} I \oint \vec{r}' \times d\vec{s}'$ $\quad d\vec{s}'$ must be along the wire so it must be along $d\vec{r}'$ – in other words

$d\vec{r}' = a(d\hat{\rho})$
$\quad + bn\cos n\phi\, d\phi\, \hat{z}$

$\vec{r}_1' + d\vec{r}' = \vec{r}_2'$ so $d\vec{r}'$ along wire

$\hat{\rho} = \cos\phi\,\hat{x} + \sin\phi\,\hat{y}$

I need $d\hat{\rho} = \cancel{\frac{\partial\hat{\rho}}{\partial\rho} d\rho} + \frac{\partial\hat{\rho}}{\partial\phi} d\phi + \cancel{\frac{\partial\hat{\rho}}{\partial z} dz}$

$\frac{\partial\hat{\rho}}{\partial\phi} = -\sin\phi\,\hat{x} + \cos\phi\,\hat{y} = \hat{\phi}$

so
$d\vec{r}' = a\,d\phi\,\hat{\phi} + \underbrace{bn\cos n\phi\, d\phi}_{dz}\,\hat{z}$ — so it is entirely in terms of $d\phi$

$\vec{M} = \frac{1}{2} I \int_0^{2\pi} (a\hat{\rho} + b\sin n\phi\,\hat{z}) \times (a\,d\phi\,\hat{\phi} + bn\cos n\phi\, d\phi\,\hat{z})$

$= \frac{1}{2} I \int_0^{2\pi} \Big( \underset{①}{a^2\hat{z}\,d\phi} + \underset{②}{abn\cos n\phi\, d\phi\,(-\hat{\phi})} + \underset{③}{ab\sin n\phi\, d\phi\,(-\hat{\rho})} \Big)$

① $2\pi a^2\hat{z}$ $\quad$ ② looks like $\int_0^{2\pi} \cos n\phi\,(\sin\phi\,\hat{x} - \cos\phi\,\hat{y})\,d\phi$

because of the orthogonality of sines/cosines, this is $\emptyset$.

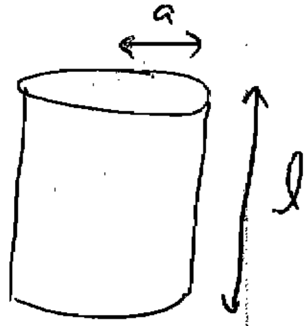
If $n=1$ then you would have $-\cos^2\phi$ which integrates to $-\pi$.

③ looks like $\int_0^{2\pi} \sin n\phi \left(-\cos\phi\, \hat{x} - \sin\phi\, \hat{y}\right) d\phi$ again $\emptyset$ by orthogonality

If $n=1$ you have $-\int \sin^2\phi\, d\phi = -\pi$ so this is $I$

$\rightarrow \vec{m} = \frac{1}{2} I \left(2\pi a^2 \hat{z}\right) \longrightarrow \boxed{\vec{m} = I \pi a^2 \hat{z}}$ times the effective area in the xy plane

so it looks the same as if you had a planar loop but if $n=1$, then the result will be different




⑤ wangsness 19-3

$\vec{m} = \frac{1}{2} \int_V \vec{r}\,' \times \vec{J}\, d\tau'$

$\vec{J} = \rho_{ch} \vec{v}$ $\quad \rho_{ch} = \frac{Q}{\pi a^2 l}$

$\vec{r}\,' = \rho' \hat{\rho} + z' \hat{z}$

$\vec{J} = \frac{Q}{\pi a^2 l} \rho' \omega\, \hat{\phi}$ $\quad \vec{m} = \frac{1}{2} \int_0^{2\pi}\int_0^{l}\int_0^{a} \left(\rho' \hat{\rho} + z' \hat{z}\right) \times \frac{Q}{\pi a^2 l} \rho' \omega\, \hat{\phi}\; \rho' d\rho' dz' d\phi'$

$= \frac{1}{2} \int_0^{2\pi}\int_0^{l^*}\int_0^{a} \left[ \frac{Q \rho'^2 \omega}{\pi a^2 l} \hat{z} + \cancel{\frac{Q z' \rho' \omega}{\pi a^2 l} (-\hat{\rho})} \right] \rho' d\rho' dz' d\phi'$

integrates to $\emptyset$ since $\int_0^{2\pi} \hat{\rho}\, d\phi = \emptyset$

$= \frac{1}{2} \frac{Q\omega}{\pi a^2 l} \hat{z}\; \frac{1}{4} a^4 (2\pi)\, l \longrightarrow \boxed{\vec{m} = \frac{Q \omega a^2}{4} \hat{z}}$ — for now $m' \hat{z}$

* this should really be $\int_{-l/2}^{l/2}$ since the cylinder is centered on the origin

now $\vec{A} = \frac{\mu_0}{4\pi} \frac{\vec{m} \times \vec{r}}{r^3} = \frac{\mu_0}{4\pi} \frac{m' \hat{z} \times \hat{r}}{r^2}$

$= \frac{\mu_0 m'}{4\pi r^2} \sin\theta\, \hat{\phi}$ $\quad$ now $\vec{B} = \vec{\nabla} \times \vec{A}$

5 - continued from inside cover $\vec{B} = \frac{\hat{r}}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta A_\phi\right) + \frac{\hat{\theta}}{r}\left(-\frac{\partial}{\partial r}(rA_\phi)\right)$

$$= \frac{\hat{r}}{r\sin\theta}\frac{\mu_0 M'}{4\pi r^2}2\sin\theta\cos\theta + \frac{\mu_0 M'}{4\pi}\sin\theta\frac{1}{r^3}\hat{\theta}$$

$\vec{M} = M\hat{z}$ at $z = z_0$

$$= \frac{\mu_0 M'}{4\pi r^3}\left(2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta}\right)$$

$$= \frac{\mu_0 Q w a^2}{16\pi r^3}\left(2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta}\right)$$

now the point dipole is far away so $\vec{B}$ dipole due to cylinder will accurately determine $\vec{F}$

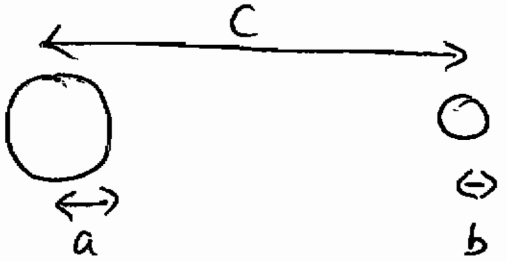
now $\vec{F} = \vec{\nabla}(\vec{m}\cdot\vec{B})$ or $\vec{F} = (\vec{m}\cdot\vec{\nabla})\vec{B}$ in this case $\vec{m} = m\hat{z}$ and $\vec{B}$ is also only along $\hat{z}$ $B_{axis} = \frac{\mu_0 Q w a^2}{8\pi r^3}\Big|_{r=z}$ so only variations along z are important

using $\vec{F} = \vec{\nabla}(\vec{m}\cdot\vec{B}) = \vec{\nabla}\left(\frac{\mu_0 m Q w a^2}{8\pi z^3}\right) = \frac{-3\mu_0 m Q w a^2}{8\pi z^4}\hat{z}$

or $\boxed{\vec{F} = \frac{-3\mu_0 m Q w a^2}{8\pi z_0^4}\hat{z}}$

force in $-\hat{z}$ since you basically have

⑥ wangsness 19-7




$M$ = flux from $B_a$ through loop b / $I_a$

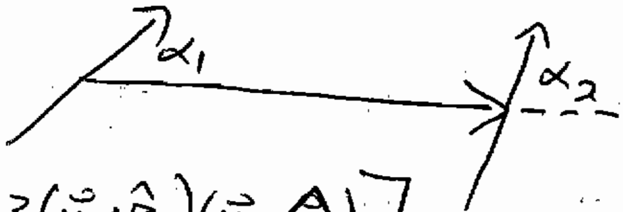
= flux from $B_b$ through loop a / $I_b$

$M = \Phi_{a\ \text{through}\ b} / I_a = B_a (\pi b^2) / I_a$ at this location $\theta = \pi/2$ so $B_r = \emptyset$

so $\vec{B}_\theta = \frac{\mu_0 m}{4\pi} \frac{\sin\theta}{r^3} \hat{\theta}$ so $B_a = \frac{\mu_0 m}{4\pi c^3}$ $m = I_a A = I_a (\pi a^2)$

$M = \frac{\mu_0}{4\pi c^3} I_a \pi a^2 (\pi b^2) / I_a \Longrightarrow \boxed{M = \frac{\mu_0 \pi a^2 b^2}{4 c^3}}$

7 wangsness 19-11




$U = \frac{\mu_0}{4\pi R^3} \left[ \vec{m}_1 \cdot \vec{m}_2 - 3(\vec{m}_1 \cdot \hat{R})(\vec{m}_2 \cdot \hat{R}) \right]$

$\alpha_1$ is fixed

$= \frac{\mu_0}{4\pi R^3} \left[ \cos(\alpha_1 - \alpha_2) m_1 m_2 - 3 m_1 m_2 \cos\alpha_1 \cos\alpha_2 \right]$

$\frac{dU}{d\alpha_2} = \emptyset = \frac{\mu_0}{4\pi R^3} \left[ (-1)\sin(\alpha_1 - \alpha_2)(-1) m_1 m_2 + 3 m_1 m_2 \cos\alpha_1 \sin\alpha_2 \right]$

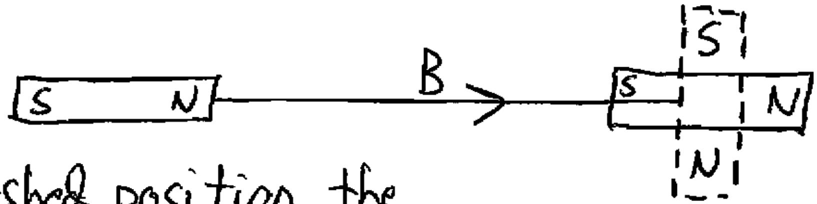
$= \sin(\alpha_1 - \alpha_2) + 3\cos\alpha_1 \sin\alpha_2 = \sin\alpha_1 \cos\alpha_2 - \sin\alpha_2 \cos\alpha_1 + 3\cos\alpha_1 \sin\alpha_2$

$\frac{\emptyset = \sin\alpha_1 \cos\alpha_2 + 2\cos\alpha_1 \sin\alpha_2}{\cos\alpha_2 \cos\alpha_1}$ $\quad \emptyset = \tan\alpha_1 + 2\tan\alpha_2$

$\Rightarrow \boxed{\tan\alpha_2 = -\frac{1}{2}\tan\alpha_1}$

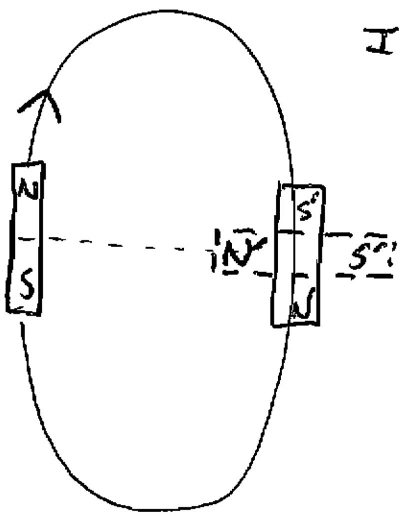
7-continued $\underline{\alpha_1 = \phi}$ $\tan\alpha_2 = -\frac{1}{2}\tan\alpha_1 = \phi$ so $\underline{\alpha_2 = \phi \text{ or } \pi}$

we basically have




If $\vec{M}_2$ were in the dashed position, the torque would take it to the solid position. so $\boxed{\alpha_2 = \phi \text{ so yes it makes sense}}$

$\underline{\alpha_1 = \pi/2}$ $\tan\alpha_2 = -\frac{1}{2}(\infty)$ $\tan\alpha_2 = -\infty$ $\underline{\alpha_2 = -\pi/2}$

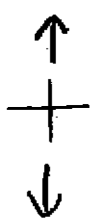



Imagine a dipole at the dashed position. N repels N' and S attracts N' so there is a CCW torque. dipole ends up as shown by solid line.

$\underline{\text{so } \alpha_2 = -\pi/2}$ ✓

when $\alpha_1$ is some other angle, it is a bit messier but $\vec{M}_2$ aligns itself with the field from $\vec{M}_1$

(8) wangsness 19-16
$$\vec{A}_{net} = \frac{\mu_0}{4\pi}\left(\frac{m\hat{z} \times \hat{R}_1}{R_1^2} + \frac{(-m\hat{z}) \times \hat{R}_2}{R_2^2}\right)$$

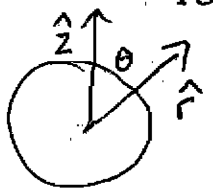


$\vec{r} = r\hat{r}$ $\vec{r}_1' = a\hat{z}$ $\vec{r}_2' = -a\hat{z}$

$\vec{R}_1 = r\hat{r} - a\hat{z}$ $\vec{R}_2 = r\hat{r} + a\hat{z}$

$$\vec{A}_{net} = \frac{\mu_0 m}{4\pi}\left(\frac{\hat{z}\times(r\hat{r}-\cancel{a\hat{z}})}{(r^2+a^2-2ar\cos\theta)^{3/2}} - \frac{\hat{z}\times(r\hat{r}+\cancel{a\hat{z}})}{(r^2+a^2+2ar\cos\theta)^{3/2}}\right)$$

$\hat{z}\times\hat{r}$ so $\hat{z}\times\hat{r} = \sin\theta\,\hat{\phi}$




$$\vec{A}_{net} = \frac{\mu_0 m r\sin\theta\,\hat{\phi}}{4\pi}\left(\frac{1}{(\ )^{3/2}} - \frac{1}{(\ )^{3/2}}\right)$$

$$= \frac{\mu_0 m r\sin\theta\,\hat{\phi}}{4\pi r^3}\left[\frac{1}{\left(1+\frac{a^2}{r^2}-\frac{2a\cos\theta}{r}\right)^{3/2}} - \frac{1}{\left(1+\frac{a^2}{r^2}+\frac{2a\cos\theta}{r}\right)^{3/2}}\right]$$

binomial expansion $a \ll r$ $\quad (1+x)^p = 1+px\ldots$

$$= \frac{\mu_0 m\sin\theta\,\hat{\phi}}{4\pi r^2}\left[\left(\cancel{1}-\cancel{\frac{3}{2}\frac{a^2}{r^2}}+\frac{3a\cos\theta}{r}\ldots\right)-\left(\cancel{1}-\cancel{\frac{3}{2}\frac{a^2}{r^2}}-\frac{3a\cos\theta}{r}\ldots\right)\right]$$

$$\rightarrow \quad \vec{A}_{net} = \frac{3\mu_0 m a\sin\theta\cos\theta\,\hat{\phi}}{2\pi r^3}$$

$\vec{B} = \vec{\nabla}\times\vec{A}$ $\quad$ let $\vec{A} = \frac{C\sin\theta\cos\theta}{r^3}\hat{\phi}$

$C = 3\mu_0 m a/2\pi$

$$\vec{B} = \frac{\hat{r}}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta A_\phi) + \frac{\hat{\theta}}{r}\left(-\frac{\partial}{\partial r}\right)(rA_\phi)$$

$$= \frac{\hat{r}}{r\sin\theta}C\frac{1}{r^3}(2\sin\theta\cos^2\theta-\sin^3\theta) - \frac{\hat{\theta}}{r}C\sin\theta\cos\theta\left(\frac{-2}{r^3}\right)$$

$2\cos^2\theta-(1-\cos^2\theta) = 3\cos^2\theta-1$

or $\boxed{\vec{B} = \frac{C}{r^4}(3\cos^2\theta-1)\hat{r} + \frac{2C\sin\theta\cos\theta}{r^4}\hat{\theta}}$

this has the exact same form as the electric quadrupole in Eq 8-55

(let $C = \frac{3Q}{16\pi\epsilon_0}$) to complete the analogy