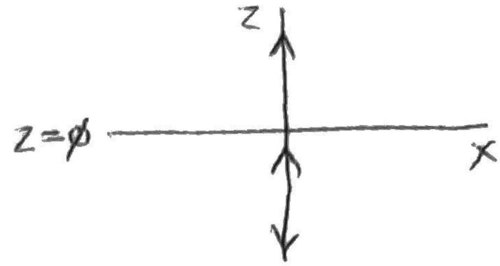
HW #4 Solutions

Reflection/Transmission - 4



1)

$$\vec{E} = E_0 e^{i(kz-\omega t)}\hat{x}$$

$$\vec{E}'' = E_0'' e^{-i(kz+\omega t)}\left(\cos\theta\,\hat{x} + \sin\theta\,\hat{y}\right)$$

$$\vec{E}' = E_0' e^{i(k'z-\omega t)}\left(\cos\phi\,\hat{x} + \sin\phi\,\hat{y}\right)$$

BCs evaluated at $z=\emptyset$, $t=\emptyset$

$\Delta\vec{E} \times \hat{n} = \emptyset$ — tangential components of $\vec{E}$ are continuous

$$E_0\hat{x} + E_0''\cos\theta\,\hat{x} + E_0''\sin\theta\,\hat{y} = E_0'\cos\phi\,\hat{x} + E_0'\sin\phi\,\hat{y} \quad (1)$$

$\Delta\vec{H} \times n = \emptyset$ — tangential components of $\vec{H}$ are continuous

$\vec{B} = \frac{1}{\omega}\vec{k}\times\vec{E}$ so $\vec{H} = \frac{1}{\omega\mu}\vec{k}\times\vec{E}$ $\quad$ ω's can be ignored - all the same

$$\frac{1}{\mu}kE_0(\hat{z}\times\hat{x}) + \frac{1}{\mu}kE_0''\left(-\hat{z}\times\hat{x}\cos\theta + (-\hat{z})\times\hat{y}\sin\theta\right) = \frac{1}{\mu'}k'E_0'\left(\cos\phi\,\hat{z}\times\hat{x} + \sin\phi\,\hat{z}\times\hat{y}\right) \quad (2)$$

from (1) $\hat{y}$ $\left(E_0''\sin\theta = E_0'\sin\phi\right)$

(2) $\hat{x}$ $\left(\frac{1}{\mu}kE_0''\sin\theta = -\frac{1}{\mu'}k'E_0'\sin\phi\right)$

so $\sin\theta/\sin\phi = \dfrac{E_0'}{E_0''} = -\dfrac{E_0'}{E_0'}\dfrac{k'}{k}\dfrac{\mu}{\mu'}$ — not possible unless $\phi = \theta = \emptyset$

$\hat{x}$ terms in (1), $\hat{y}$ terms in (2) not needed for argument

) from class $\frac{E_0'}{E_0} = \frac{2n}{n+n'}$, $\frac{E_0''}{E_0} = \frac{n-n'}{n+n'}$

$\langle S \rangle = \langle u \rangle v = \frac{1}{2}\epsilon E^2 \frac{1}{\sqrt{\mu\epsilon}} = \frac{1}{2}\sqrt{\frac{\epsilon}{\mu}}\, E^2$ $\quad n = \frac{c}{v} = c\sqrt{\mu\epsilon}$

$\sqrt{\epsilon} = \frac{n}{c\sqrt{\mu}}$

$\langle S \rangle = \frac{1}{2}\frac{n}{c\mu} E^2$

$$\frac{\langle S\rangle_R}{\langle S\rangle_I} = \frac{\langle S''\rangle}{\langle S\rangle} = \frac{1/2\, n/c\mu \left(\frac{n-n'}{n+n'}\right)^2 E_0^2}{1/2\, n/c\mu\, E_0^2} = \left(\frac{n-n'}{n+n'}\right)^2 = \frac{1}{2}$$

so $n-n' = \pm\frac{1}{\sqrt{2}}(n+n')$ $\quad n\left(1 \mp \frac{1}{\sqrt{2}}\right) = n'\left(1 \pm \frac{1}{\sqrt{2}}\right)$

$$\frac{n}{n'} = \frac{(\sqrt{2}\pm 1)/\sqrt{2}}{(\sqrt{2}\mp 1)/\sqrt{2}} = \frac{\sqrt{2}\pm 1}{\sqrt{2}\mp 1} = 5.83 \text{ and } .172$$

)

these are inverses of each other

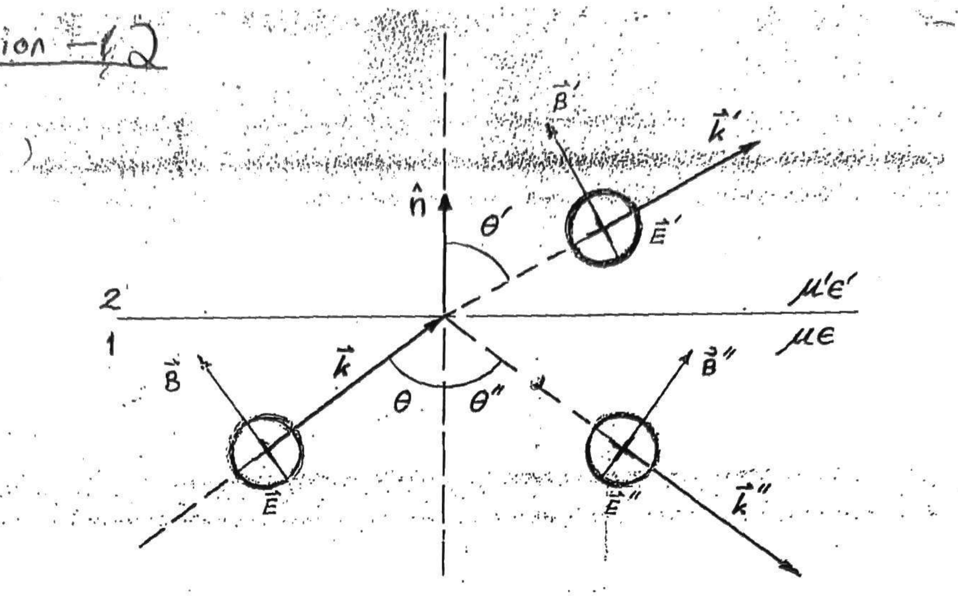
so $\frac{1}{2}$ is reflected, $\frac{1}{2}$ is transmitted for light incident from either side of the interface

you can check but transmitted fraction is also $\frac{1}{2}$

)

Reflection/Transmission – 2

Energy Conservation



$\underline{\vec{E} \parallel (\hat{n} \times \vec{k})}$

$\vec{E}_\perp$ **case**

$$\langle \vec{S} \rangle = \langle u \rangle v \, \hat{k}$$

$$= \frac{1}{2}\sqrt{\frac{\epsilon}{\mu}}\, E_0^2 \, \hat{k}$$

$$v = \frac{c}{n} = \frac{1}{\sqrt{\mu\epsilon}}$$

so $n = c\sqrt{\mu\epsilon}$ so

$$\langle \vec{S} \rangle = \frac{1}{2}\frac{n}{\mu c} E_0^2 \, \hat{k}$$

$$|\vec{S}\cdot\hat{n}| = |S'\cdot\hat{n}| + |S''\cdot\hat{n}|$$

Ignore $\frac{1}{2}$, μ, and c in $\vec{S}$

$$\frac{E_0'}{E_0} = \frac{2n\cos\theta}{n\cos\theta + \frac{\mu}{\mu'}\sqrt{n'^2 - n^2\sin\theta}} = \frac{2n\cos\theta}{a+b}$$

$$\frac{E_0''}{E_0} = \frac{a-b}{a+b} \qquad \mu = \mu'$$

$$n E_0^2 \cos\theta = n\left(\frac{a-b}{a+b}\right)^2 \cos\theta\, E_0^2 + n' \frac{4n^2\cos^2\theta}{(a+b)^2} E_0^2 \cos\theta'$$

cancel $n, E_0^2, \cos\theta$

$$(a+b)^2 = (a-b)^2 + 4nn'\cos\theta\cos\theta'$$

$$a^2 + 2ab + b^2 = a^2 - 2ab + b^2 + 4nn'\cos\theta\cos\theta' \qquad 4ab = 4nn'\cos\theta\cos\theta'$$

$$b = \sqrt{n'^2 - n^2\sin^2\theta} = \sqrt{n'^2 - n'^2\sin^2\theta'} = n'\sqrt{1-\sin^2\theta'} = n'\cos\theta'$$

$$n\cos\theta\,(n'\cos\theta') = nn'\cos\theta\cos\theta' \checkmark$$

Wangsness 25-3 $\tan \theta_p = \frac{n_2}{n_1}$ $\sin\theta_c = \frac{n_2}{n_1}$

so we need to prove that $\theta_p - \theta_c < \emptyset$

or $\tan^{-1}\frac{n_2}{n_1} - \sin^{-1}\frac{n_2}{n_1} < \emptyset$ $\tan^{-1}\frac{n_2}{n_1} < \sin^{-1}\frac{n_2}{n_1}$

now take tan of both sides

"y" $= n_2$

"r" $= n_1$

so "x" $= \sqrt{n_1^2 - n_2^2}$

$\frac{n_2}{n_1} < \frac{n_2}{\sqrt{n_1^2-n_2^2}}$

so yes $\theta_p < \theta_c$

so as you increase θ from $\emptyset$, you reach the angle at which the reflected light is completely polarized $\perp$ to the plane of incidence before you reach the angle that results in total internal reflection

$n_2/n_1 = .75 \Rightarrow \theta_p = 36.9^\circ$, $\theta_c = 48.6^\circ$

$.5$ $\theta_p = 26.6^\circ$, $\theta_c = 30^\circ$

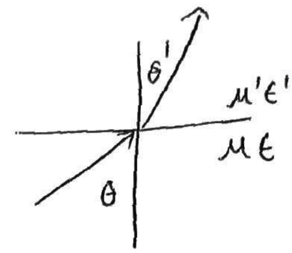
$.25$ $\theta_p = 14.0^\circ$, $\theta_c = 14.5^\circ$

as $\theta_p \to \emptyset$, $\theta_c \to \theta_p$ (since $\sin\theta \simeq \tan\theta \simeq \theta$)

$(n_1 \gg n_2)$

Reflection/Transmission - 1

At Brewster angle, $\mu = \mu'$



$n' > n$

a) $$\frac{E_0'}{E_0} = \frac{2nn'\cos\theta}{\frac{\mu}{\mu'} n'^2 \cos\theta + n\sqrt{n'^2 - n^2\sin^2\theta}}\Bigg|_{\theta=\theta_p}$$

$\div$ by $\cos\theta_p$

$$\frac{E_0'}{E_0} = \frac{2nn'}{n'^2 + n\sqrt{\frac{n'^2 - n^2\sin^2\theta}{\cos^2\theta}}} = \frac{2nn'}{n'^2 + n\sqrt{\frac{n'^2}{\cos^2\theta} - n^2\, n'^2/n^2}}$$

$$= \frac{2nn'}{n'^2 + nn'\sqrt{\frac{1-\cos^2\theta}{\cos^2\theta}}} = \frac{2nn'}{n'^2 + nn'\tan\theta_p} = \frac{2nn'}{n'^2 + nn'\, n'/n} = \frac{2nn'}{2n'^2}$$

$$\rightarrow \boxed{\frac{E_0'}{E_0}\bigg|_{\parallel} = \frac{n}{n'}}$$

$\langle u \rangle = \frac{1}{2}\epsilon E_0^2$

b)

$u = \frac{1}{2}\epsilon E_0^2 \qquad u' = \frac{1}{2}\epsilon' \frac{n^2}{n'^2} E_0^2$

$n = \frac{c}{v} = c\sqrt{\mu\epsilon}$

so $n^2 \propto \epsilon$

so $u = \frac{1}{2} n^2 E_0^2$, $\quad u' = \frac{1}{2} n'^2 \frac{n^2}{n'^2} E_0^2 = \frac{1}{2} n^2 E_0^2$ ✓

so both waves have the same energy density even though the electric fields have different magnitudes

c) $\langle S \rangle = \langle u \rangle v$ so definitely not. $v \propto \frac{1}{n}$

the energy densities are the same but the energy is transported at different rates and also in different directions

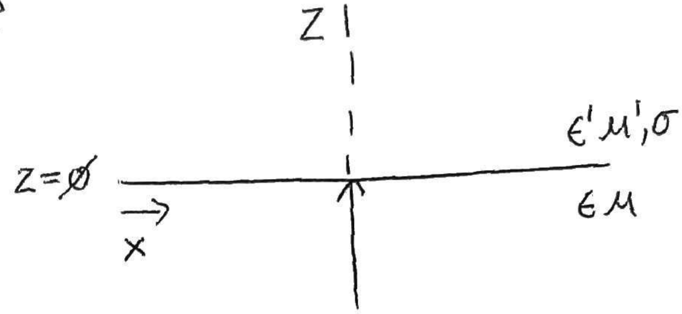
so $$\frac{|\vec{S}'|}{|\vec{S}|} = \frac{\frac{1}{2}n^2E_0^2\frac{1}{n'}}{\frac{1}{2}n^2E_0^2\frac{1}{n}} = \frac{n}{n'}$$

Reflection/Transmission - 3

a)

$$\vec{E} = E_0 e^{i(kz-\omega t)} \hat{x}$$

$$\vec{E}'' = -E_0 e^{-i(kz+\omega t)} \hat{x}$$



$$\vec{E}_{total} = \vec{E} + \vec{E}'' = E_0 \hat{x} \left(e^{i(kz-\omega t)} - e^{-i(kz+\omega t)} \right)$$

$$= E_0 \hat{x} \left[\cos(kz-\omega t) \pm \cos(kz+\omega t) + i\left(\sin(kz-\omega t) + \sin(kz+\omega t)\right)\right]$$

$$\text{physical wave} = Re\left(\vec{E}_{total}\right) = E_0 \hat{x} \left(\cos(kz-\omega t) - \cos(kz+\omega t)\right)$$

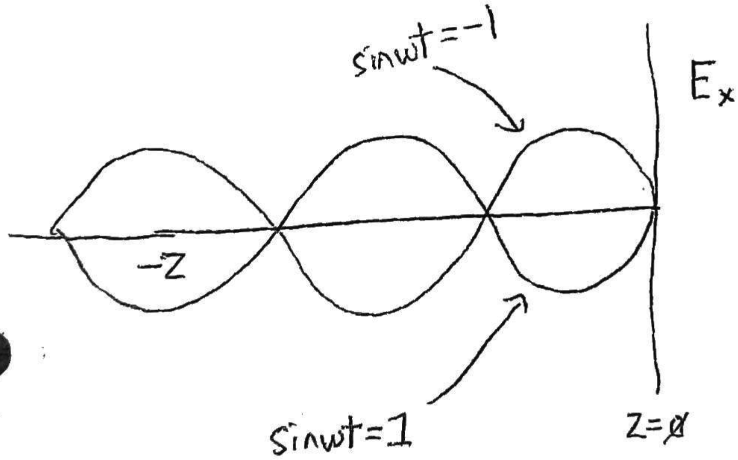
$$\cos(a \pm b) = \cos a \cos b \mp \sin a \sin b$$

$$\Rightarrow E_0 \hat{x} \left(\cos kz \cos \omega t + \sin kz \sin \omega t - (\cos kz \cos \omega t - \sin kz \sin \omega t)\right)$$

$$\rightarrow \boxed{\vec{E}_{real} = 2E_0 \sin kz \sin \omega t \, \hat{x}}$$

standing wave with a node at the boundary ($z=\emptyset$) since E, E'' differ in phase by π

the vacuum exists for $z < \emptyset$ so



and naturally in the good conductor limit, there is no field for $z > \emptyset$

$\omega t =$ ~~...~~ $\frac{\pi}{2}, \frac{5\pi}{2}$, etc

$\underline{\hat{n}\cdot(\vec{D}_2-\vec{D}_1)=\sigma}$ $\quad \hat{n}\cdot\left(\vec{E}_0{}' - (\vec{E}_0+\vec{E}_0{}'')\right) = \sigma$

$\vec{E}_0{}' = \emptyset$ since its an ideal conductor $\quad \vec{E}_0 + E_0'' \cdot \hat{n} = \emptyset$

since E in $\hat{x}$ and $\hat{n}$ is $\hat{z}$ $\longrightarrow$ $\underline{\sigma = \emptyset}$

(no normal component on either side of interface)

c) $\vec{n}\times(\vec{H}_2-\vec{H}_1) = \vec{K}_f \quad \vec{H}_2 = H_0' = \emptyset \quad \hat{n}\times\left(\vec{H}_0+\vec{H}_0{}''\right) = -\vec{K}$

$\vec{B} = \dfrac{\vec{K}\times\vec{E}}{\omega}$ so $\vec{H} = \dfrac{1}{\omega\mu}\vec{K}\times\vec{E} \quad \vec{H}_0 = \dfrac{K}{\omega\mu}\,\hat{z}\times E_0\, e^{i(kz-\omega t)}\,\hat{x}$

$\vec{H}_0{}'' = \dfrac{K}{\omega\mu}(-\hat{z})\times(-E_0)e^{-i(kz+\omega t)}\hat{x} \qquad = \dfrac{K E_0}{\omega\mu} e^{i(kz-\omega t)}\,\hat{y} = \sqrt{\dfrac{\epsilon}{\mu}}E_0\, e^{i(kz-\omega t)}$

$= \sqrt{\dfrac{\epsilon}{\mu}}E_0\, e^{-i(kz+\omega t)}\,\hat{y}$ *

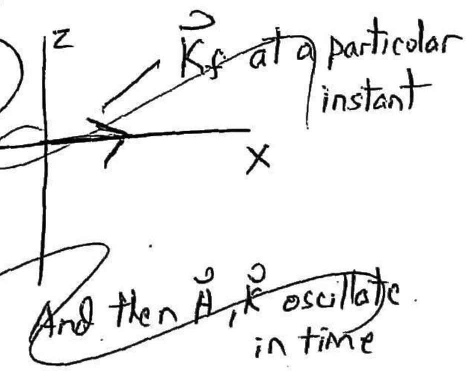
~~so Real$(\vec{H}_0+\vec{H}_0{}'') = 2\sqrt{\frac{\epsilon}{\mu}}E_0\, e^{i(kz+\omega t)}\,\hat{y}\cos kz$~~

~~so $\hat{n}\times 2\sqrt{\frac{\epsilon}{\mu}}E_0\cos(kz+\omega t)\,\hat{y} = \vec{K}_f \quad 2\sqrt{\frac{\epsilon}{\mu}}\, e^{i(kz+\omega t)}\,\hat{z}\times\hat{y} = -K_f$~~

~~$\rightarrow \vec{K}_f = 2\sqrt{\frac{\epsilon}{\mu}}\, e^{i(kz+\omega t)}\,\hat{x}$ now take the real part and set $z=\emptyset$~~

~~$\Rightarrow \vec{K}_f = 2\sqrt{\frac{\epsilon}{\mu}}E_0\cos\omega t\,\hat{x}$~~

see correction on next page



~~at that instant, $\vec{H}$ is clearly in $\hat{y}$ for $z<\emptyset$ so its consistent. And then $\vec{H}, \vec{k}$ oscillate in time~~

* so $\vec{H}_0{}'' = \vec{H}_0$, they must be the same. $\vec{E}$ flipped in phase by π so $\vec{H}$ couldn't flip or $\vec{S}$ would be in the wrong direction

<u>correction</u> $\hat{n} \times (\vec{H}_o + \vec{H}_o'') = -\vec{K}_f$

$$\vec{H}_o + \vec{H}_o'' = \left[\sqrt{\tfrac{\epsilon}{\mu}}\, E_o\, e^{i(kz-\omega t)} + \sqrt{\tfrac{\epsilon}{\mu}}\, E_o\, e^{-i(kz+\omega t)}\right]\hat{y}$$

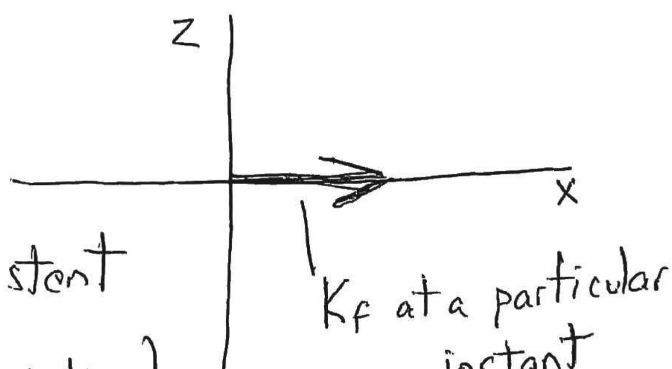
now $z=\emptyset$ so $\vec{H}_o + \vec{H}_o'' = \sqrt{\tfrac{\epsilon}{\mu}}\, E_o \left[e^{-i\omega t} + e^{-i\omega t}\right]\hat{y}$

since we want the real $\vec{K}_f$, lets take $Re(\vec{H}_o + \vec{H}_o'')$ $= 2\sqrt{\tfrac{\epsilon}{\mu}}\, E_o \cos\omega t\, \hat{y}$

$$\hat{n} \times 2\sqrt{\tfrac{\epsilon}{\mu}}\, E_o \cos\omega t\, \hat{y} = 2\sqrt{\tfrac{\epsilon}{\mu}}\, E_o \cos\omega t\, \hat{z} \times \hat{y} = -\vec{K}_f$$

$$\Rightarrow \underline{\vec{K}_f = 2\sqrt{\tfrac{\epsilon}{\mu}}\, E_o \cos\omega t\, \hat{x}}$$

at the instant shown, $\vec{H}$ is clearly in $\hat{y}$ for $z<\emptyset$ – so it's consistent



(And then $\vec{H}$ and $\vec{K}_f$ oscillate in time)