ELECTROMAGNETIC THEORY

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Problem A: Show that an arbitrary function $f(x)$ can be expanded on the interval $0 \leq x \leq a$ in a modified Fourier-Bessel series

$$f(x) = \sum_{n=1}^{\infty} A_n J_\nu \left(y_{\nu n} \frac{x}{a} \right) ,$$

where

$$A_n = \frac{2}{a^2 \left(1 - \frac{\nu^2}{{y_{\nu n}}^2}\right) \, {J_\nu}^2(y_{\nu n})} \int_0^a f(x) \, x \, J_\nu \left(y_{\nu n} \frac{x}{a} \right) \, dx \, .$$

Problem B: (a) Show that for a system of current-carrying elements in empty space, the total energy in the magnetic field is

$$W = \frac{1}{2c^2} \int d^3x \, \int d^3x' \, \frac{\mathbf{J(x)} \cdot \mathbf{J(x')}}{|\mathbf{x} - \mathbf{x'}|} \, ,$$

where $\mathbf{J(x)}$ is the current density.

(b) If the current configuration consists of n circuits carrying currents I_1, I_2, I_3,..., I_n, show that the energy can be expressed as

$$W = \frac{1}{2} \sum_{i=1}^{n} L_i \, {I_i}^2 + \sum_{i=1}^{n} \sum_{j>i}^{n} M_{ij} I_i I_j \, .$$

Exhibit integral expressions for the self-inductances (L_i) and the mutual inductances (M_{ij}).

Problem C: Consider two current loops (as in Fig. 5.3 of Jackson) whose orientation in space is fixed, but whose relative separation can be changed. Let O_1 and O_2 be origins in the two loops, fixed relative to each loop, and $\mathbf{x}_1$ and $\mathbf{x}_2$ be coordinates of elements $d\mathbf{l}_1$ and $d\mathbf{l}_2$, respectively, of the loops referred to the respective origins. Let R be the relative coordinate of the origins, directed from loop 2 to loop 1.

(a) Starting from the expression for the force between the loops, show that it can be written

$$\mathbf{F}_{12} = I_1 I_2 \vec{\nabla}_R M_{12}(\mathbf{R}) \, ,$$

where M_{12} is the mutual inductance of the loops,

$$M_{12}(\mathbf{R}) = \frac{1}{c^2} \oint \oint \frac{d\mathbf{l}_1 \cdot d\mathbf{l}_2}{|\mathbf{x}_1 - \mathbf{x}_2 + \mathbf{R}|} ,$$

and it is assumed that the orientation of the loops does not change with $\mathbf{R}$.

(b) Show that the mutual inductance, viewed as a function of $\mathbf{R}$, is a solution of the Laplace equation,

$$\nabla_R^2 M_{12}(\mathbf{R}) = 0 .$$

The importance of this result is that the uniqueness of solutions of the Laplace equation allows the exploitation of the properties of such solutions, provided a solution can be found for a particular value of $\mathbf{R}$.