Problem 1

The relationships between the event co-ordinates must be linear in order to guarantee uniqueness in reciprocal correspondences between the conclusions of equivalent observers. Thus, using the notation

$$\begin{aligned} x_0 &= ict \\ x_1 &= x \\ x_2 &= y \\ x_3 &= z \end{aligned} \qquad \left(\text{so } s^2 = -\sum_{\mu=0}^{3} x_\mu^2 \right) \tag{1}$$

we have

$$x_\mu' = \sum_{\nu=0}^{3} a_{\mu\nu} x_\nu \tag{2}$$

such that

$$\sum_{\mu=0} (x_\mu')^2 = \sum_{\nu=0} x_\nu^2 \tag{3}$$

for a light path, e.g. The problem is now to find the 16 elements $a_{\mu\nu}$ of the matrix transformation.

Since

$$\sum_\mu (x_\mu')^2 = \sum_{\nu,\rho} x_\nu x_\rho \sum_\mu a_{\mu\nu} a_{\mu\rho} = \sum_\nu x_\nu^2 ,$$

the 16 coefficients must satisfy

$$\sum_{\mu=0}^{3} a_{\mu\nu} a_{\mu\rho} = \delta_{\nu\rho} \qquad \nu, \rho = 0,1,2,3 \quad , \qquad (4)$$

(the so-called orthonormality condition). This provides 10 independent equations, since (4) is symmetric in the interchange $\nu \Leftrightarrow \rho$. Besides the 4 equations with $\delta_{\nu\nu} = 1$, there are 6 independent off-diagonal equations with $\delta_{\nu \neq \rho} = \delta_{\rho \neq \nu} = 0$.

Since the primed and unprimed axes of the two frames are parallel (by choice here), we can't have projections on each other:

$$a_{i \neq j} = 0 \qquad i,j = 1,2,3 \qquad (5)$$

This eliminates 6 of the coefficients, leaving 10.

But now the three off-diagonal equations with $ij = 12, 23, 31$ reduce to

$$\sum_{\mu=0}^{3} a_{\mu i}\, a_{\mu j} = a_{0i}\, a_{0j} = 0 , \tag{6}$$

so that at least 2 of the 3 coefficients a_{01}, a_{02}, a_{03} must also vanish. The remaining 3 off-diagonal equations yield

$$\sum_{\mu=0}^{3} a_{\mu i}\, a_{\mu 0} = a_{ii}\, a_{i0} + a_{0i}\, a_{00} = 0 . \qquad (i = 1, 2, 3) \tag{7}$$

Thus,

$$a_{03} = - a_{30} \frac{a_{33}}{a_{00}} , \tag{8}$$

$$\text{and} \quad a_{01} = a_{02} = a_{10} = a_{20} = 0 . \tag{9}$$

The diagonal equations reduce to

$$a_{11}^2 = a_{22}^2 = 1 \tag{10}$$

$$a_{33}^2 + a_{03}^2 = 1 \tag{11}$$

$$a_{30}^2 + a_{00}^2 = 1 . \tag{12}$$

If we now take all $a_{\nu\nu} > 0$ to give like senses to variables increasing in parallel, we solve (8), (10), (11) and (12) to give

$$a_{11} = a_{22} = +1 \tag{13}$$

$$a_{00} = a_{33} \quad (>0) \tag{14}$$

Thus,

$$x' = x \tag{15}$$

$$y' = y \quad . \tag{16}$$

Also,

$$a_{03} = -a_{30}$$

$$a_{30}^2 = a_{03}^2 = 1 - a_{33}^2 \quad . \tag{17}$$

Thus,

$$z' = a_{33} z + a_{30} ict \tag{18}$$

$$t' = a_{33} t + \frac{iz}{c} a_{30} \quad . \tag{19}$$

But the origins are $z = vt$ apart, according to O', so

$$0 = a_{33} vt + a_{30} ict \tag{20}$$

$$\therefore \quad a_{30} = i\beta a_{33} \tag{21}$$

$(\beta \equiv \frac{v}{c})$. Thus,

$$-\beta^2 a_{33}^2 = 1 - a_{33}^2$$

$$\therefore \quad a_{33}^2 = \frac{1}{1-\beta^2} \equiv \gamma^2. \qquad (22)$$

So finally, we arrive at the Lorentz transformation

$$x' = x$$
$$y' = y$$
$$z' = \gamma(z - vt)$$
$$t' = \gamma\left(t - \frac{vz}{c^2}\right)$$
$$\gamma \equiv \frac{1}{\sqrt{1-(v/c)^2}}.$$

Jackson 11.3 :

Arrange the axes so that $\vec{v}_1$ and $\vec{v}_2$ are along x_1.

Then,

$$x_0'' = \gamma_2 (x_0' - \beta_2 x_1')$$
$$x_1'' = \gamma_2 (x_1' - \beta_2 x_0')$$
$$x_2'' = x_2$$
$$x_3'' = x_3 .$$

That is,

$$x_0'' = \gamma_2 \gamma_1 (x_0 - \beta_1 x_1) - \gamma_2 \gamma_1 \beta_2 (x_1 - \beta_1 x_0)$$
$$x_1'' = \gamma_2 \gamma_1 (x_1 - \beta_1 x_0) - \gamma_2 \gamma_1 \beta_2 (x_0 - \beta_1 x_1).$$

$$\therefore \quad x_0'' = (\gamma_2 \gamma_1 + \gamma_2 \gamma_1 \beta_2 \beta_1) x_0 - (\gamma_2 \gamma_1 \beta_1 + \gamma_2 \gamma_1 \beta_2) x_1$$
$$= \gamma_2 \gamma_1 \left\{ (1 + \beta_2 \beta_1) x_0 - (\beta_1 + \beta_2) x_1 \right\}$$
$$x_1'' = \gamma_2 \gamma_1 \left\{ (1 + \beta_2 \beta_1) x_1 - (\beta_1 + \beta_2) x_0 \right\}$$

$$\therefore \quad x_0'' = \gamma_2 \gamma_1 (1 + \beta_2 \beta_1) \left(x_0 - \frac{\beta_1 + \beta_2}{1 + \beta_2 \beta_1} x_1 \right)$$
$$x_1'' = \gamma_2 \gamma_1 (1 + \beta_2 \beta_1) \left(x_1 - \frac{\beta_1 + \beta_2}{1 + \beta_2 \beta_1} x_0 \right).$$

Write these as

$$x_0'' = \bar{\gamma}\,(x_0 - \bar{\beta}\, x_1)$$
$$x_1'' = \bar{\gamma}\,(x_1 - \bar{\beta}\, x_0)$$

where
$$\bar{\gamma} \equiv \frac{1}{\sqrt{1-\bar{\beta}^2}} .$$

Then,
$$1-\bar{\beta}^2 = \frac{(1-\beta_1^2)(1-\beta_2^2)}{(1+\beta_1\beta_2)^2}$$

is consistent with
$$\bar{\beta} = \frac{\beta_1+\beta_2}{1+\beta_1\beta_2}$$

$$\therefore \quad \frac{\bar{v}}{c} = \frac{v_1+v_2}{c + v_1\, v_2/c}$$

$$\therefore \quad \bar{v} = \frac{v_1+v_2}{1+\frac{v_1 v_2}{c^2}} .$$

Jackson 11.5:

We know that

$$\vec{u}_{\parallel} = \frac{\vec{u}_{\parallel}' + \vec{v}}{1 + \frac{\vec{v}\cdot\vec{u}'}{c^2}} \qquad (1)$$

$$\vec{u}_{\perp} = \frac{\vec{u}_{\perp}'}{\gamma\left(1 + \frac{\vec{v}\cdot\vec{u}'}{c^2}\right)} \quad , \qquad (2)$$

and that $$x_0 = \gamma(x_0' + \vec{\beta}\cdot\vec{x}'). \qquad (3)$$

Now, $$du_{\parallel} = \frac{du_{\parallel}' + dv}{1 + \frac{v u_{\parallel}'}{c^2}} - \frac{u_{\parallel}' + v}{\left(1 + \frac{v u_{\parallel}'}{c^2}\right)^2} \cdot \frac{v}{c^2} du_{\parallel}'$$

$$= \frac{du_{\parallel}'\left(1 - \frac{v^2}{c^2}\right)}{\left(1 + \frac{v u_{\parallel}'}{c^2}\right)^2} . \qquad (4)$$

Therefore,

$$c\frac{du_{\parallel}}{dx_0} \equiv a_{\parallel} = c\,\frac{\gamma^{-2}\,du_{\parallel}'}{\left(1 + \frac{v u_{\parallel}'}{c^2}\right)^2} \cdot \frac{\gamma^{-1}}{dx_0' + \vec{\beta}\cdot d\vec{x}'}$$

$$= \frac{c}{\gamma^3}\,\frac{du_{\parallel}'}{dx_0'}\,\frac{1}{\left(1 + \frac{v u_{\parallel}'}{c^2}\right)^2}\,\frac{1}{1 + \vec{\beta}\cdot\frac{d\vec{x}'}{dx_0'}}$$

$$= a_{\parallel}'\,\frac{\left(1 - \frac{v^2}{c^2}\right)^{3/2}}{\left(1 + \frac{v u_{\parallel}'}{c^2}\right)^3} \qquad (5)$$

Similarly,

$$d\vec{u}_\perp = \frac{d\vec{u}_\perp'}{\gamma(1+\frac{\vec{v}\cdot\vec{u}'}{c^2})} + \frac{\vec{u}_\perp'}{\gamma}\frac{-1}{(1+\frac{vu_\parallel'}{c^2})^2}\frac{v}{c^2}du_\parallel'$$

$$= \frac{1}{\gamma(1+\frac{vu_\parallel'}{c^2})^2}\left\{ d\vec{u}_\perp' + d\vec{u}_\perp' \frac{vu_\parallel'}{c^2} - du_\parallel' \frac{v\vec{u}_\perp'}{c^2}\right\}$$

Therefore,

$$c\frac{d\vec{u}_\perp}{dx_0} \equiv \vec{a}_\perp =$$

$$\frac{1}{\gamma^2}\frac{1}{(1+\frac{vu_\parallel'}{c^2})^2}\left\{\frac{c\frac{d\vec{u}_\perp'}{dx_0'} + c\frac{d\vec{u}_\perp'}{dx_0'}\frac{vu_\parallel'}{c^2} - c\frac{du_\parallel'}{dx_0}\frac{v\vec{u}_\perp'}{c^2}}{1+\vec{\beta}\cdot\frac{d\vec{x}'}{dx_0}}\right\}$$

$$= \frac{(1-\frac{v^2}{c^2})}{(1+\frac{vu_\parallel'}{c^2})^3}\left\{\vec{a}_\perp' + \vec{a}_\perp'\frac{\vec{v}\cdot\vec{u}'}{c^2} - \vec{u}_\perp'\frac{\vec{a}'\cdot\vec{v}}{c^2}\right\}$$

Since $\quad \vec{A}\times(\vec{B}\times\vec{C}) = (\vec{A}\cdot\vec{C})\vec{B} - (\vec{A}\cdot\vec{B})\vec{C}$,

$$\frac{1}{c^2}(\vec{v}\cdot\vec{u}')\vec{a}_\perp' - \frac{1}{c^2}(\vec{v}\cdot\vec{a}')\vec{u}_\perp' =$$

$$\frac{1}{c^2}\left\{(\vec{v}\cdot\vec{u}')\vec{a}_\perp' - (\vec{v}\cdot\vec{a}_\perp')\vec{u}_\parallel' + (\vec{v}\cdot\vec{u}_\perp')\vec{a}_\parallel' - (\vec{v}\cdot\vec{a}_\parallel)\vec{u}_\perp'\right\}$$

$$= \frac{1}{c^2}\vec{v}\times(\vec{a}'\times\vec{u}') \qquad \text{since } \vec{v}\cdot\vec{a}_\perp' = \vec{v}\cdot\vec{u}_\perp' = 0$$

Thus,

$$\vec{a}_\perp = \frac{(1-\frac{v^2}{c^2})}{(1+\frac{\vec{v}\cdot\vec{u}'}{c^2})^3}\left\{\vec{a}_\perp' + \frac{\vec{v}}{c^2}\times(\vec{a}'\times\vec{u}')\right\}.$$

Jackson 11-9

(a) $$x'^{\alpha} = (g^{\alpha\beta} + \varepsilon^{\alpha\beta})\, g_{\beta\gamma}\, (g^{\gamma\varepsilon} + \varepsilon'^{\gamma\varepsilon})\, x'_{\varepsilon}$$

Since the x^{α} are linearly independent,

$$(g^{\alpha\beta} + \varepsilon^{\alpha\beta})\, g_{\beta\gamma}\, (g^{\gamma\varepsilon} + \varepsilon'^{\gamma\varepsilon}) = g^{\alpha\varepsilon}$$

$$\therefore \quad g^{\alpha\beta} g_{\beta\gamma} g^{\gamma\varepsilon} + g^{\alpha\beta} g_{\beta\gamma} \varepsilon'^{\gamma\varepsilon} +$$

$$\varepsilon^{\alpha\beta} g_{\beta\gamma} g^{\gamma\varepsilon} + \varepsilon^{\alpha\beta} g_{\beta\gamma} \varepsilon'^{\gamma\varepsilon} = g^{\alpha\varepsilon}$$

$$\therefore \quad \delta^{\alpha}{}_{\gamma}\, g^{\gamma\varepsilon} + \delta^{\alpha}{}_{\gamma}\, \varepsilon'^{\gamma\varepsilon} + \varepsilon^{\alpha\beta}\, \delta_{\beta}{}^{\varepsilon} + g_{\beta\gamma}\, \varepsilon^{\alpha\beta} \varepsilon'^{\gamma\varepsilon} = g^{\alpha\varepsilon}$$

$$\therefore \quad \varepsilon'^{\alpha\varepsilon} + \varepsilon^{\alpha\varepsilon} + g_{\beta\gamma}\, \varepsilon^{\alpha\beta} \varepsilon'^{\gamma\varepsilon} = 0$$

Since $\varepsilon^{\alpha\beta}$ is infinitesimal, we omit higher orders:

$$\varepsilon'^{\alpha\varepsilon} = -\varepsilon^{\alpha\varepsilon}$$

(b) $$x'^{\alpha} x'_{\alpha} = x^{\alpha} x_{\alpha}$$

$$(g^{\alpha\beta} + \varepsilon^{\alpha\beta})\, g_{\alpha\gamma}\, (g^{\gamma\delta} + \varepsilon^{\gamma\delta})\, x_{\beta} x_{\delta} = x^{\alpha} x_{\alpha}$$

$$\therefore \quad (g^{\alpha\beta} + \varepsilon^{\alpha\beta})\, g_{\alpha\gamma}\, (g^{\gamma\delta} + \varepsilon^{\gamma\delta}) = g^{\delta\beta}$$

$$\therefore \quad g^{\beta\delta} + \varepsilon^{\beta\delta} + \varepsilon^{\delta\beta} + g_{\alpha\gamma}\, \varepsilon^{\alpha\beta} \varepsilon^{\gamma\delta} = g^{\delta\beta}$$

$$\therefore \quad \varepsilon^{\beta\delta} + \varepsilon^{\delta\beta} + g_{\alpha\gamma}\,\varepsilon^{\alpha\beta}\varepsilon^{\gamma\delta} = 0 .$$

Again, since $\varepsilon^{\alpha\beta}$ is infinitesimal, we consider this to first order only:

$$\varepsilon^{\beta\delta} = -\,\varepsilon^{\delta\beta}$$

(c)

$$x'^{\alpha} = (g^{\alpha\beta} + \varepsilon^{\alpha\beta})\,g_{\beta\gamma}\,x^{\gamma}$$

$$= (\delta^{\alpha}{}_{\gamma} + g_{\beta\gamma}\,\varepsilon^{\alpha\beta})\,x^{\gamma}$$

$$\therefore \quad x' = A x$$

where $A^{\alpha}{}_{\gamma} = \delta^{\alpha}{}_{\gamma} + g_{\beta\gamma}\,\varepsilon^{\alpha\beta} = \delta^{\alpha}{}_{\gamma} + \varepsilon^{\alpha}{}_{\gamma}$

Since $\varepsilon^{\alpha\beta}$ is infinitesimal, we can write

$$A \approx I + \varepsilon + \varepsilon^{2} \dots$$

$$= e^{+\varepsilon}$$

$$\therefore \quad \varepsilon \approx L .$$

Jackson 11.10

(a)
$$\vec{\varepsilon}\cdot\vec{S} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -\varepsilon_3 & \varepsilon_2 \\ 0 & \varepsilon_3 & 0 & -\varepsilon_1 \\ 0 & -\varepsilon_2 & \varepsilon_1 & 0 \end{pmatrix}$$

$$(\vec{\varepsilon}\cdot\vec{S})^2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -\varepsilon_2^2-\varepsilon_3^2 & \varepsilon_1\varepsilon_2 & \varepsilon_1\varepsilon_3 \\ 0 & \varepsilon_1\varepsilon_2 & -\varepsilon_1^2-\varepsilon_3^2 & \varepsilon_2\varepsilon_3 \\ 0 & \varepsilon_1\varepsilon_3 & \varepsilon_2\varepsilon_3 & -\varepsilon_1^2-\varepsilon_2^2 \end{pmatrix}$$

$$(\vec{\varepsilon}\cdot\vec{S})^3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & (\varepsilon_1^2\varepsilon_3+\varepsilon_3^3+\varepsilon_2^2\varepsilon_3) & (-\varepsilon_2\varepsilon_3^2-\varepsilon_1^2\varepsilon_2-\varepsilon_2^3) \\ 0 & (-\varepsilon_2^2\varepsilon_3-\varepsilon_3^3-\varepsilon_1^2\varepsilon_3) & 0 & (\varepsilon_1\varepsilon_3^2+\varepsilon_1^3+\varepsilon_1\varepsilon_2^2) \\ 0 & (\varepsilon_2^2+\varepsilon_2\varepsilon_3^2+\varepsilon_1^2\varepsilon_2) & (-\varepsilon_1\varepsilon_2^2-\varepsilon_1^3-\varepsilon_1\varepsilon_3^2) & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \varepsilon_3 & -\varepsilon_2 \\ 0 & -\varepsilon_3 & 0 & \varepsilon_1 \\ 0 & \varepsilon_2 & -\varepsilon_1 & 0 \end{pmatrix}$$

$$= -\vec{\varepsilon}\cdot\vec{S}$$

as long as $\vec{\varepsilon}$ is a unit vector.

$$\vec{\varepsilon}' \cdot \vec{K} = \left(\begin{array}{c|ccc} 0 & \varepsilon_1' & \varepsilon_2' & \varepsilon_3' \\ \hline \varepsilon_1' & & & \\ \varepsilon_2' & & 0 & \\ \varepsilon_3' & & & \end{array}\right)$$

$$(\vec{\varepsilon}' \cdot \vec{K})^2 = \left(\begin{array}{c|ccc} 1 & & \vec{0} & \\ \hline & \varepsilon_1'^2 & \varepsilon_1'\varepsilon_2' & \varepsilon_1'\varepsilon_3' \\ \vec{0} & \varepsilon_2'\varepsilon_1' & \varepsilon_2'^2 & \varepsilon_2'\varepsilon_3' \\ & \varepsilon_3'\varepsilon_1' & \varepsilon_3'\varepsilon_2' & \varepsilon_3'^2 \end{array}\right)$$

$$(\vec{\varepsilon}' \cdot \vec{K})^3 = \left(\begin{array}{c|ccc} 0 & \varepsilon_1' & \varepsilon_2' & \varepsilon_3' \\ \hline \varepsilon_1' & & & \\ \varepsilon_2' & & 0 & \\ \varepsilon_3' & & & \end{array}\right)$$

$$= \vec{\varepsilon}' \cdot \vec{K}$$

since $\quad \varepsilon_1'^3 + \varepsilon_2'^2 \varepsilon_1' + \varepsilon_3'^2 \varepsilon_1' = \varepsilon_1' \left(\varepsilon_1'^2 + \varepsilon_2'^2 + \varepsilon_3'^2\right)$

$= \varepsilon_1'$

etc.

(b)
$$e^{-\zeta \hat{\beta}\cdot\vec{K}} = I + (-\zeta\hat{\beta}\cdot\vec{K}) + \frac{1}{2!}(-\zeta\hat{\beta}\cdot\vec{K})^2 + \cdots$$

$$= I - \zeta\hat{\beta}\cdot\vec{K} + \frac{1}{2}\zeta^2(\hat{\beta}\cdot\vec{K})^2 - \frac{1}{3!}\zeta^3(\hat{\beta}\cdot\vec{K})^3 \cdots$$

$$= I - \zeta\hat{\beta}\cdot\vec{K} - \frac{1}{3!}\zeta^3\hat{\beta}\cdot\vec{K} - \frac{1}{5!}\zeta^5\hat{\beta}\cdot\vec{K} \cdots\cdots$$
$$+ \frac{1}{2}\zeta^2(\hat{\beta}\cdot\vec{K})^2 + \frac{1}{4!}\zeta^4(\hat{\beta}\cdot\vec{K})^2 \cdots\cdots$$

$$= I - \hat{\beta}\cdot\vec{K}\left(\zeta + \frac{1}{3!}\zeta^3 + \frac{1}{5!}\zeta^5 \cdots\right)$$
$$+ (\hat{\beta}\cdot\vec{K})^2\left(-1 + 1 + \frac{1}{2}\zeta^2 + \frac{1}{4!}\zeta^4 \cdots\right)$$

$$= I - \hat{\beta}\cdot\vec{K}\sinh\zeta + (\hat{\beta}\cdot\vec{K})^2(\cosh\zeta - 1).$$