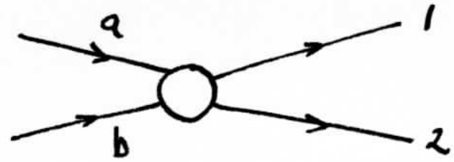
Solutions for HW #2

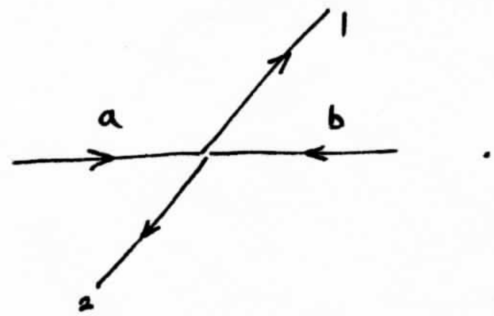
Problem 1 :

(a)



$$
\begin{aligned}
s+t+u &= p_a^2 + p_b^2 + 2p_a\cdot p_b + p_a^2 + p_1^2 \\
&\quad - 2p_a\cdot p_1 + p_b^2 + p_1^2 - 2p_b\cdot p_1 \\
&= p_a^2 + p_b^2 + p_1^2 + \{ p_a^2 + p_b^2 + p_1^2 + 2p_a\cdot p_b \\
&\qquad - 2p_a\cdot p_1 - 2p_b\cdot p_1 \} \\
&= p_a^2 + p_b^2 + p_1^2 + (p_a + p_b - p_1)^2 \\
&= p_a^2 + p_b^2 + p_1^2 + p_2^2 \\
&= (m_a^2 + m_b^2 + m_1^2 + m_2^2)c^2 \equiv C
\end{aligned}
$$

(b) In the CMS frame, we have



Now, $t = (p_a - p_1)^2$

$$= (m_a^2 + m_1^2)c^2 - \frac{2}{c^2} E_a^* E_1^* + 2\vec{P}_a^* \cdot \vec{P}_1^*$$

(where $*$ ≡ cms frame)

$$\therefore \quad t = (m_a^2 + m_1^2)c^2 - \frac{2E_a^* E_1^*}{c^2} + 2 P_a^* P_1^* \cos\theta^* . \tag{1}$$

But $\qquad s = (E_a^* + E_b^*)^2/c^2 ,$

where $\qquad E_a^* = (p_a^{*2} c^2 + m_a^2 c^4)^{1/2} .$

Thus, $\qquad s = p_a^{*2} + m_a^2 c^2 + p_b^{*2} + m_b^2 c^2 + 2E_a^* E_b^*/c^2$

$$\therefore \quad s + m_a^2 c^2 - m_b^2 c^2 = p_a^{*2} + p_b^{*2} + 2m_a^2 c^2 + 2E_a^* E_b^*/c^2$$

Since $\qquad p_a^* = p_b^* ,$

$$s + m_a^2 c^2 - m_b^2 c^2 = 2(E_a^* + E_b^*)E_a^*/c^2$$

$$\therefore \quad E_a^* = \frac{s + m_a^2 c^2 - m_b^2 c^2}{2\sqrt{s}} \cdot c \tag{2}$$

Similarly, $$E_b^* = \frac{s + m_b^2 c^2 - m_a^2 c^2}{2\sqrt{s}} \cdot c \tag{3}$$

Also,
$$2p_a^{*2} = 2p_b^{*2} = s - m_a^2c^2 - m_b^2c^2 - \frac{2}{c^2}E_a^*E_b^*$$
$$= s - m_a^2c^2 - m_b^2c^2 - 2\frac{(s+m_a^2c^2-m_b^2c^2)(s+m_b^2c^2-m_a^2c^2)}{4s}$$
$$= \{2s^2 - 2s(m_a^2c^2 + m_b^2c^2) - s^2 + 2s(m_a^2c^2 + m_b^2c^2) - 2m_a^2c^2m_b^2c^2 + m_a^4c^4 + m_b^4c^4\} \div 2 \times s$$

$$\therefore \quad p_a^* = p_b^* = \frac{\lambda^{1/2}(s, m_a^2c^2, m_b^2c^2)}{2\sqrt{s}} \qquad (4)$$

Thus,
$$t = (m_a^2 + m_1^2)c^2 - \frac{1}{2s}\{(s + m_a^2c^2 - m_b^2c^2)(s + m_1^2c^2 - m_2^2c^2)c^2 - \cos\theta^* . \lambda^{1/2}(s, m_a^2c^2, m_b^2c^2)\,\lambda^{1/2}(s, m_1^2c^2, m_2^2c^2)\}$$

(5)

All the quantities on the rhs are constant for a given s, except for θ^*. Thus,

$$t_{min} = (m_a^2+m_1^2)c^2 - \frac{1}{2s}\{(s+m_a^2c^2-m_b^2c^2)(s+m_1^2c^2-m_2^2c^2) + \lambda^{1/2}(s, m_a^2c^2, m_b^2c^2)\,\lambda^{1/2}(s, m_1^2c^2, m_2^2c^2)$$

with $\underline{\cos\theta^* = -1}$, $\theta^* = \pi$ (backward)

$$t_{max} = (m_a^2 + m_1^2)c^2 - \frac{1}{2s}\{(s + m_a^2c^2 - m_b^2c^2)(s + m_1^2c^2 - m_2^2c^2) - \lambda^{1/2}(s, m_a^2c^2, m_b^2c^2)\,\lambda^{1/2}(s, m_1^2c^2, m_2^2c^2)\}$$

with $\underline{\cos\theta^* = +1}$, $\theta^* = 0$ (forward)

To find u, we may use $s + t + u = \sum_i m_i^2c^2$.

Then,

$$u = (m_a^2 + m_b^2 + m_1^2 + m_2^2)c^2 - s - t$$

$$\therefore \quad u_{max} = -s + (m_b^2 + m_2^2)c^2 + \frac{1}{2s}\{(s + m_a^2c^2 - m_b^2c^2)(s + m_1^2c^2 - m_2^2c^2) + \lambda^{1/2}(s, m_a^2c^2, m_b^2c^2)\,\lambda^{1/2}(s, m_1^2c^2, m_2^2c^2)\}$$

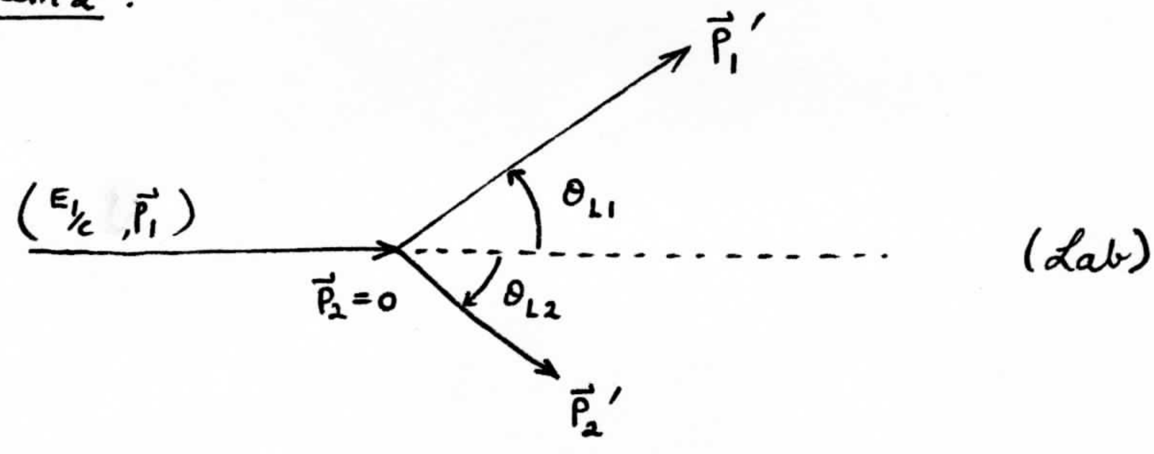
with $\underline{\cos\theta^* = -1}$, $\theta^* = \pi$ (backward)

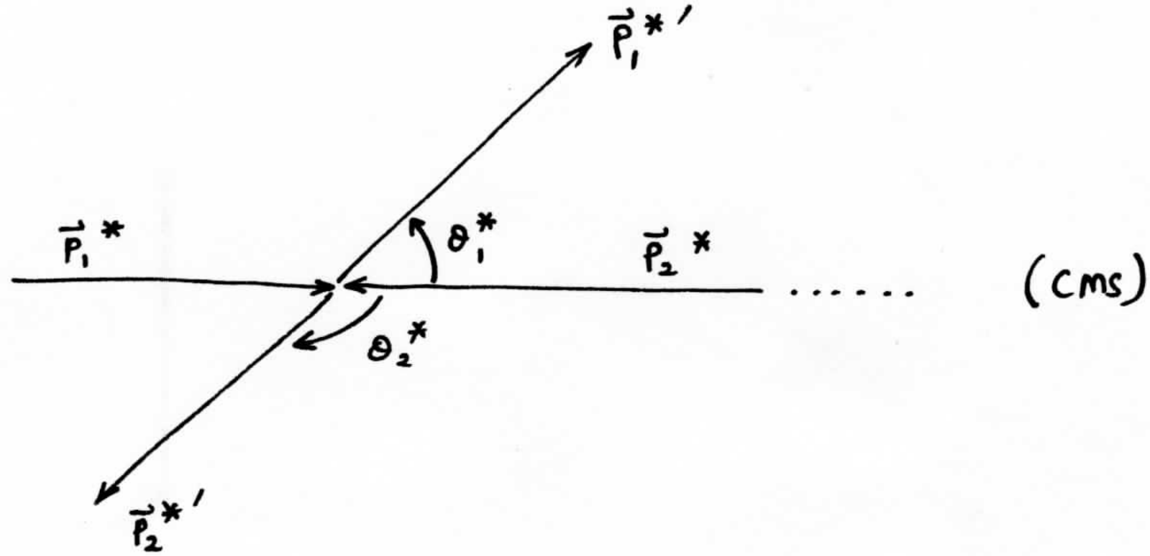
and

$$u_{min} = -s + (m_b^2 + m_2^2)c^2 + \frac{1}{2s}\{(s + m_a^2c^2 - m_b^2c^2)(s + m_1^2c^2 - m_2^2c^2) - \lambda^{1/2}(s, m_a^2c^2, m_b^2c^2)\,\lambda^{1/2}(s, m_1^2c^2, m_2^2c^2)\}$$

with $\underline{\cos\theta^* = +1}$, $\theta^* = 0$ (forward).

Problem 2 :





Let $c^2 S = E_0^2$ be the invariant energy squared. Then,

$$E_0^2 = (E_1 + m_2 c^2)^2 - \vec{P}_1^2 c^2$$

$$\therefore \quad E_0^2 = m_1^2 c^4 + m_2^2 c^4 + 2 m_2 c^2 E_1 \qquad (1)$$

The invariant angle-type variable is the invariant momentum transfer

$$t = (p_2 - p_2')^2$$

$$= (E_2 - E_2')^2/c^2 - (\vec{P}_2 - \vec{P}_2')^2$$

$$= E_2^2/c^2 + E_2'^2/c^2 - 2E_2E_2'/c^2 - (\vec{P}_2')^2$$

$$= E_2^2/c^2 - 2E_2E_2'/c^2 + m_2^2c^2$$

$$= 2m_2^2c^2 - 2E_2E_2'/c^2$$

$$= 2m_2^2c^2 - 2m_2E_2'$$

Since $E_2' = K_2 + m_2c^2$, we can write

$$t = 2m_2^2c^2 - 2m_2^2c^2 - 2m_2K_2$$

$$\therefore \boxed{t = -2m_2K_2}. \qquad (2)$$

In cms, we have

$$|\vec{P}_1^*| = |\vec{P}_2^*|$$

$$s = (E_1^* + E_2^*)^2/c^2$$

$$= m_1^2c^2 + m_2^2c^2 + 2m_2E_1$$

$$\therefore \quad \sqrt{s} = \sqrt{\vec{p}_1^{*2} + m_1^2 c^2} + \sqrt{\vec{p}_2^{*2} + m_2^2 c^2}$$

$$\therefore \quad s = \vec{p}_1^{*2} + m_1^2 c^2 + \vec{p}_2^{*2} + m_2^2 c^2 + 2 E_1^* E_2^* / c^2$$

$$= 2(\vec{p}_1^*)^2 + m_1^2 c^2 + m_2^2 c^2 + 2 E_1^* E_2^* / c^2$$

$$\therefore \quad s + m_1^2 c^2 - m_2^2 c^2 = 2 E_1^{*2} / c^2 + 2 E_1^* E_2^* / c^2$$

$$\therefore \quad E_1^* = \frac{s + m_1^2 c^2 - m_2^2 c^2}{2\sqrt{s}} \cdot c \qquad (3)$$

Thus,

$$E_1^* = E_1^{*\prime} = \frac{s + m_1^2 c^2 - m_2^2 c^2}{2\sqrt{s}} \cdot c$$

and

$$E_2^* = E_2^{*\prime} = \frac{s - m_1^2 c^2 + m_2^2 c^2}{2\sqrt{s}} \cdot c \qquad (4)$$

Similarly (as was shown in Problem 1),

$$|\vec{p}_1^*| = |\vec{p}_1^*|' = |\vec{p}_2^*| = |\vec{p}_2^*|'$$

$$= \frac{\lambda^{1/2}(s, m_1^2 c^2, m_2^2 c^2)}{2\sqrt{s}} . \qquad (5)$$

Now, $$t = 2m_2^2c^2 - 2E_2^* E_2^{*\prime}/c^2 + 2|P_2^*||P_2^{*\prime}| \cos\theta_2^* /c^2$$

$$\therefore \quad \cos\theta_2^* = \frac{t - 2m_2^2c^2 + 2E_2^*E_2^{*\prime}/c^2}{2|\vec{P}_2^*||\vec{P}_2^{*\prime}|/c^2}$$

$$\therefore \quad \cos\theta_2^* = \left\{2s(t - m_2^2c^2 - m_2^2c^2) + (s+m_2^2c^2 - m_1^2c^2)(s+m_2^2c^2-m_1^2c^2)\right\} \div \lambda(s, m_2^2c^2, m_1^2c^2)/c^2$$

$$\therefore \quad \cos\theta_2^* = 1 + \frac{2st}{\lambda(s, m_2^2c^2, m_1^2c^2)}c^2$$

$$\therefore \quad t = -\frac{2}{(c\sqrt{s})^2}\,\frac{1}{2}\,\lambda(s, m_1^2c^2, m_2^2c^2)\cdot\sin^2\tfrac{1}{2}\theta.$$

But $$s = m_1^2c^2 + m_2^2c^2 + 2m_2cE_1$$

$$\therefore \quad \left\{\vec{P}_1^2c^2 + m_1^2c^4\right\}^{1/2} = \frac{s - m_1^2c^2 - m_2^2c^2}{2m_2c}$$

$$\therefore \quad \vec{P}_1^2c^2 = \frac{(s - m_1^2c^2 - m_2^2c^2)^2}{(2m_2c)^2} - m_1^2c^4$$

$$\therefore \quad \vec{P}_1 = \frac{\lambda^{1/2}(s, m_1^2c^2, m_2^2c^2)}{2m_2c^2}$$

Thus,

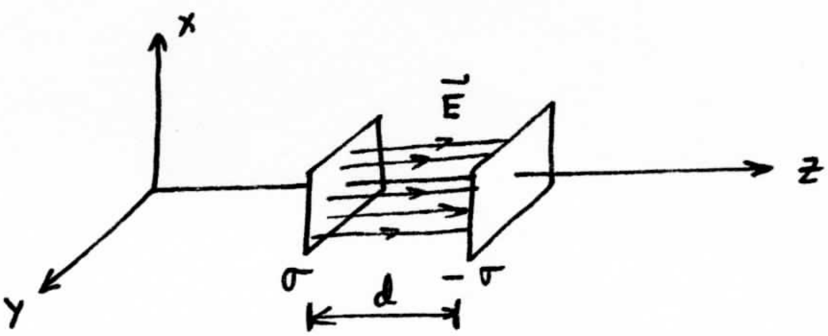
$$-2m_2 K_2 = -\frac{2}{c^2 s} \cdot \frac{1}{2} \vec{P}_1^{\,2} \cdot 4m_2^2 c^4 . \sin^2 \tfrac{1}{2}\theta_2^*$$

$$\therefore K_2 = \frac{c^2 \vec{P}_1^{\,2}\, 4 m_2^2 c^4}{m_2 c^2\, 2 s c^2} \sin^2 \tfrac{1}{2}\theta_2^*$$

$$\therefore \boxed{K_2 = \frac{2 m_2 \vec{P}_1^{\,2} c^4}{E_0^{\,2}} \cdot \sin^2 \frac{\theta_2^*}{2}} .$$

Problem 3 :

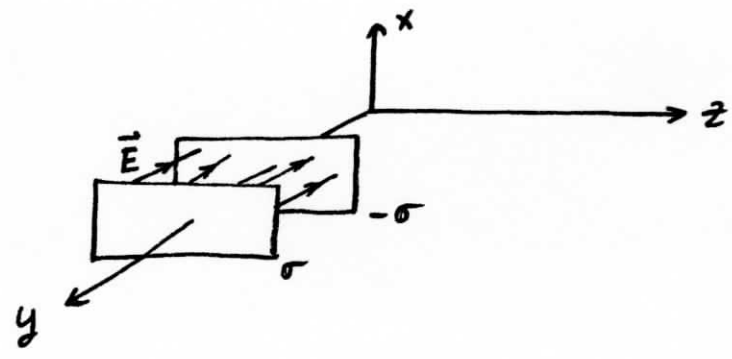
(a)



In frame K', the capacitor is moving with velocity v and the plates are separated by d/γ. The surface charge density is unchanged $\sigma' = \sigma$, because the net charge on a surface element is invariant, and the surface area of the element is also invariant, because y and x components are unchanged. Since the field depends only on surface charge density and not on plate separation, we have $E' = E$, so that in general

$$\vec{E}'_{\parallel} = \vec{E}_{\parallel} . \qquad (1)$$

(b)



The charge density σ is now increased by a factor γ because of length contraction, and we also have a surface current density (along the plates) of magnitude $\mu' = \sigma' v$, which gives rise to a magnetic field in the x-direction of magnitude $B'_x = -\frac{4\pi}{c}\mu'$.

Thus, for this case, we have

$$\vec{E}'_\perp = \gamma \vec{E}_\perp$$

$$\vec{B}'_\perp = -\gamma \vec{\beta} \times \vec{E}_\perp \,.$$

Note that it is also possible to treat the case of an initially pure magnetic field by a similar model.

Problem 4 :

(a) We must find a tensor expression that reduces to $\vec{J} = \sigma \vec{E}$ in the fluid rest frame. We know that the "time components" F^{ok} contain the electric field. We also know that the fluid 4-velocity u^α has only time components in the fluid rest frame. Thus we try

$$J^\alpha = \frac{\sigma}{c} F^{\alpha\beta} u_\beta . \qquad (1)$$

This equation has the right space components in the rest frame

$$J^k = \frac{\sigma}{c} F^{k\nu} U_\nu = \sigma E^k, \qquad (2)$$

but the time component $(\alpha = 0)$ in the rest frame gives $\rho = 0$, an unacceptable constraint. Thus, we want to subtract out of (1) its time component in the rest frame, that is, we need to project out only that part which is orthogonal to $\vec{U}$:

$$J^{\alpha} - \frac{1}{c^2} J^{\beta} U_{\beta} U^{\alpha} = \frac{\sigma}{c} F^{\alpha\nu} U_{\nu} , \qquad (3)$$

where we have used $F^{\alpha\nu} U_{\alpha} U_{\nu} = 0$.

This is manifestly a tensor equation. Its space components give $\vec{J} = \sigma \vec{E}$ in the rest frame (where $U^k = 0$) and its time component in the rest frame gives $0 = 0$, that is, no constraint on ρ.

(b)
$$J^{k} = \frac{\sigma}{c} F^{k\nu} U_{\nu} + \frac{1}{c^2} J^{\beta} U_{\beta} U^{k}$$

$$= \frac{\sigma}{c} \left\{ \gamma c E^{k} - \gamma \epsilon^{kij} B_i v_j \right\}$$

$$+ \frac{1}{c^2} \gamma v^{k} \left\{ \rho c \gamma c - \gamma \vec{J} \cdot \vec{v} \right\}$$

$$= \gamma \sigma \left[E^{k} + (\vec{\beta} \times \vec{B})^{k} \right]$$

$$+ \gamma^2 \rho v^{k} - \gamma^2 \beta^{k} \vec{J} \cdot \vec{\beta} \qquad (4)$$

$$\therefore \quad \vec{J} = \gamma \sigma \left[\vec{E} + \vec{\beta} \times \vec{B} \right] + \gamma^2 \rho \vec{v} - \gamma^2 \vec{\beta}\, \vec{J} \cdot \vec{\beta}$$

Thus,

$$\vec{J}\cdot\vec{\beta} = \gamma\sigma\left[\vec{E}\cdot\vec{\beta} + \cancelto{0}{(\vec{\beta}\times\vec{B})\cdot\vec{\beta}}\right] + \gamma^2\rho\vec{v}\cdot\vec{\beta} - \gamma^2\vec{\beta}^2\,\vec{J}\cdot\vec{\beta}$$

$$\therefore\ \vec{J}\cdot\vec{\beta} = \frac{\gamma\sigma\,\vec{E}\cdot\vec{\beta}}{1+\gamma^2\beta^2} + \frac{\gamma^2\rho\,\vec{v}\cdot\vec{\beta}}{1+\gamma^2\beta^2}$$

But $\quad 1+\gamma^2\beta^2 = 1 + \dfrac{\beta^2}{1-\beta^2} = \gamma^2.$

Thus,

$$\vec{J}\cdot\vec{\beta} = \frac{\sigma\,\vec{E}\cdot\vec{\beta}}{\gamma} + \rho\,\vec{v}\cdot\vec{\beta}.$$

Therefore,

$$\vec{J} = \gamma\sigma\left[\vec{E} + \vec{\beta}\times\vec{B}\right] + \gamma^2\rho\vec{v} - \gamma^2\vec{\beta}\,\rho\vec{v}\cdot\vec{\beta} - \gamma^2\vec{\beta}\,\frac{\sigma\vec{E}\cdot\vec{\beta}}{\gamma}$$

$$= \gamma\sigma\left[\vec{E} + \vec{\beta}\times\vec{B} - \vec{\beta}(\vec{\beta}\cdot\vec{E})\right] + \rho\vec{v}$$

(c)
$$\rho' c = 0 = \gamma (\rho c - \vec{\beta} \cdot \vec{J})$$

$$\therefore \quad \rho = \frac{1}{c} \vec{\beta} \cdot \vec{J}$$

Thus,
$$\vec{J} = \gamma \sigma [\vec{E} + \vec{\beta} \times \vec{B} - \vec{\beta} (\vec{\beta} \cdot \vec{E})] + \vec{\beta} \, \vec{\beta} \cdot \vec{J} .$$

Again,
$$\vec{\beta} \cdot \vec{J} = \gamma \sigma [\vec{E} \cdot \vec{\beta} - \beta^2 \, \vec{E} \cdot \vec{\beta}] + \beta^2 \, \vec{\beta} \cdot \vec{J}$$

$$\therefore \quad \vec{\beta} \cdot \vec{J} = \frac{\gamma \sigma \, \vec{E} \cdot \vec{\beta} \, (1 - \beta^2)}{1 - \beta^2}$$

$$= \gamma \sigma \, \vec{E} \cdot \vec{\beta} ,$$

So
$$\boxed{\vec{J} = \gamma \sigma (\vec{E} + \vec{\beta} \times \vec{B})} .$$

Problem 5 :

(a)

$$\vec{E}_{\parallel}' = \vec{E}_{\parallel} \qquad (1)$$

$$\vec{B}_{\parallel}' = \vec{B}_{\parallel} \qquad (2)$$

$$\vec{E}_{\perp}' = \gamma\left(\vec{E}_{\perp} + \vec{\beta}\times\vec{B}\right) \qquad (3)$$

$$\vec{B}_{\perp}' = \gamma\left(\vec{B}_{\perp} - \vec{\beta}\times\vec{E}\right) \qquad (4)$$

Add (1) and (3) , and (2) and (4) :

$$\vec{E}' = \vec{E}_{\parallel} + \gamma\left(\vec{E}_{\perp} + \vec{\beta}\times\vec{B}\right) \qquad (5)$$

$$\vec{B}' = \vec{B}_{\parallel} + \gamma\left(\vec{B}_{\perp} - \vec{\beta}\times\vec{E}\right) . \qquad (6)$$

Since $\vec{E}_{\parallel} = \vec{B}_{\parallel} = 0$ here , we can write for the entire fields in the drifting frame

$$\vec{E}' = \gamma\left(\vec{E} + \frac{\vec{v}\times\vec{B}}{c}\right) \qquad (7)$$

$$\vec{B}' = \gamma\left(\vec{B} - \frac{\vec{v}\times\vec{E}}{c}\right) . \qquad (8)$$

Now, $$\gamma = \frac{1}{\sqrt{1-\beta^2}}$$

$$= \frac{1}{\sqrt{1-\left|\frac{\vec{E}\times\vec{B}}{B^2}\right|^2}} = \frac{1}{\sqrt{1-\frac{E^2}{B^2}}} .$$

Thus, substituting for $\vec{v}$ in (7) & (8),

$$\vec{E}' = \gamma \left(\vec{E} + \frac{\vec{E}\times\vec{B}}{B^2} \times \vec{B} \right) = 0 \qquad (9)$$

$$\vec{B}' = \gamma \left(\vec{B} - \frac{\vec{E}\times\vec{B}}{B^2} \times \vec{E} \right)$$

$$= \gamma \vec{B} \left(1 - \frac{E^2}{B^2} \right)$$

$$= \vec{B} \left(1 - \frac{E^2}{B^2} \right)^{1/2} . \qquad (10)$$

As in the NR case, the electric force is reduced to zero relative to the <u>drifting</u> frame. This is just the advantage provided by reference to it. The magnetic field is not changed in direction but is <u>weakened</u> in magnitude.

(b) With only a uniform magnetic field $\vec{B}' = \vec{B}/\gamma$ existing in the drifting frame, the equation of motion with respect to it is

$$\frac{d\vec{p}'}{dt'} = q\,\vec{u}' \times \frac{\vec{B}}{\gamma c}\,, \tag{11}$$

where

$$\vec{p}' = \gamma' m\, \vec{u}'\,, \tag{12}$$

$$\gamma' = \frac{1}{\sqrt{1-(u'/c)^2}}\,. \tag{13}$$

The transverse motion of the particle describes a circle of radius R', where

$$|\dot{\vec{u}}'| = \frac{u_\perp'^{\,2}}{R'}$$

and $u_\perp'$ is the tangential velocity. From (11), we have

$$\dot{\vec{u}}' = q\,\frac{\vec{u}' \times \vec{B}}{\gamma' \gamma c m}\,,$$

so that

$$R' = \frac{\gamma' \gamma m c\, u_\perp'}{q B}\,. \tag{14}$$

The angular velocity of the circle is

$$\omega_0' = \frac{u_\perp'}{R'} = \frac{qB}{\gamma'\gamma mc} .$$

(c) The motion as observed in the laboratory will not differ in quality from the cycloidal orbiting found in the NR case, but the cycloids will be distorted by Lorentz contractions in space and by dilations of the time scale.

When $\vec{u} = 0$, the velocity transformation yields $\vec{u}' = -\vec{v}$ for the initial velocity in the drifting frame, and then its magnitude stays constant at the value $u' = u_\perp' = |\vec{v}|$ thereafter.

With $\gamma' = \gamma$ here,

$$R' = \frac{mc^2}{q} \frac{E}{B^2 - E^2} \tag{15}$$

which is larger than the value

$$R = \frac{mc^2}{q} \frac{E}{B}$$

found in the NR approximation because of the weakening of the magnetic field relative to the drifting frame.

Since the dimensions of the orbit $\perp$ to the drift are not altered by transformation to the lab frame, the orbital diameter is now

$$2R' = 2\gamma^2 R .$$

Correspondingly, the drift during each cycle of the motion is

$$\Delta z = |\vec{v}|\, T ,$$

where $T = \gamma T' = \gamma \frac{2\pi}{\omega_0'}$. That is,

$$\Delta z = 2\pi \gamma R' = 2\pi\gamma^3 R .$$

Thus, each of the cycles in space is <u>enlarged</u> by a factor γ^2 in the $\perp$ direction and by γ^3 in the drift direction.