Solutions for HW #3

Problem 1:

(a) The solution to this problem requires some tensor index manipulation plus use of Maxwell's equations in tensor form.

$$T^{\mu}_{\ \mu} = g_{\mu\nu} T^{\mu\nu}$$

$$= \frac{g_{\mu\nu}}{4\pi} F^{\mu\alpha} F^{\nu}_{\ \alpha} - \frac{g_{\mu\nu}}{16\pi} g^{\mu\nu} F^{\alpha\beta} F_{\alpha\beta}$$

$$= \frac{F^{\mu\alpha} F_{\mu\alpha}}{4\pi} - \frac{4}{16\pi} F^{\alpha\beta} F_{\alpha\beta}$$

$$= 0$$

(b) $$\partial_\nu T^{\mu\nu} = \frac{1}{4\pi} \Big\{ (\partial_\nu F^{\mu\alpha}) F^{\nu}_{\ \alpha} + F^{\mu\alpha} \partial_\nu F^{\nu}_{\ \alpha}$$

$$- \frac{1}{4} g^{\mu\nu} (\partial_\nu F^{\alpha\beta} \cdot F_{\alpha\beta} + F^{\alpha\beta} \cdot \partial_\nu F_{\alpha\beta}) \Big\}$$

$$= \frac{1}{4\pi} \Big\{ g_{\nu\delta} \partial^{\delta} F^{\mu\alpha} \cdot g^{\nu\lambda} F_{\lambda\alpha} +$$

$$g^{\alpha\delta} F^{\mu}_{\ \delta} \partial_\nu F^{\nu\lambda} g_{\alpha\lambda}$$

$$- \frac{1}{4} (\partial^{\mu} F^{\alpha\beta} \cdot F_{\alpha\beta} + F^{\alpha\beta} \cdot \partial^{\mu} F_{\alpha\beta}) \Big\}$$

$$= \frac{1}{4\pi}\Big\{ \partial^\nu F^{\mu\alpha} \cdot F_{\nu\alpha} - F^\mu{}_\alpha \cdot \partial_\nu F^{\alpha\nu}$$

$$- \tfrac{1}{2}\partial^\mu F^{\alpha\beta} \, F_{\alpha\beta} \Big\}$$

$$= \frac{1}{4\pi}\Big\{ - F^\mu{}_\alpha \cdot \partial_\nu F^{\alpha\nu} - F_{\alpha\beta}\big(\partial^\beta F^{\mu\alpha}$$

$$+ \tfrac{1}{2}\partial^\mu F^{\alpha\beta} \Big\}$$

But $\qquad F_{\alpha\beta}\,\partial^\beta F^{\mu\alpha} = F_{\beta\alpha}\,\partial^\alpha F^{\mu\beta} = F_{\alpha\beta}\,\partial^\alpha F^{\beta\mu}$

(relabeling indices and using antisymmetry of $F^{\alpha\beta}$).

Thus,

$$\partial_\nu T^{\mu\nu} = \frac{1}{4\pi}\Big\{ - F^\mu{}_\alpha \cdot \partial_\nu F^{\alpha\nu} - \tfrac{1}{2} F_{\alpha\beta}\big(\partial^\beta F^{\mu\alpha} + \partial^\alpha F^{\beta\mu} + \partial^\mu F^{\alpha\beta}\big)\Big\}.$$

In free space, Maxwell's equations give

$$\partial_\nu F^{\alpha\nu} = 0$$

$$\partial^\beta F^{\mu\alpha} + \partial^\alpha F^{\beta\mu} + \partial^\mu F^{\alpha\beta} = 0$$

so that $\qquad \partial_\nu T^{\mu\nu} = 0.$

problem 2

$$\tilde{L} = -\frac{1}{2} m u_\alpha u^\alpha - \frac{q}{c} u_\alpha A^\alpha$$

The action integral is

$$I = \int_{\tau_1}^{\tau_2} \tilde{L}\, d\tau .$$

Thus, the Euler-Lagrange equations are

$$\frac{d}{d\tau}\left(\frac{\partial \tilde{L}}{\partial u^\mu}\right) - \frac{\partial \tilde{L}}{\partial x^\mu} = 0$$

Therefore

$$\frac{\partial \tilde{L}}{\partial u^\mu} = -m u_\mu - \frac{q}{c} A_\mu$$

$$\frac{\partial \tilde{L}}{\partial x^\mu} = -\frac{q}{c} u_\alpha \frac{\partial}{\partial x^\mu} A^\alpha$$

Thus

$$\frac{d}{d\tau}\left(m u_\mu + \frac{q}{c} A_\mu\right) - \frac{q}{c} u_\alpha \frac{\partial}{\partial x^\mu} A^\alpha = 0$$

$$u^\alpha \left[\frac{\partial}{\partial x^\alpha}\left(m u_\mu + \frac{q}{c} A_\mu\right) - \frac{q}{c}\frac{\partial}{\partial x^\mu} A_\alpha\right] = 0$$

$$u^\alpha \left[\frac{\partial}{\partial x^\alpha} m u_\mu + \frac{q}{c}\left(A_{\mu,\alpha} - A_{\alpha,\mu}\right)\right] = 0$$

$$\frac{d}{d\tau} m u^\mu = \frac{q}{c} u_\alpha F^{\mu\alpha}$$

$$\mu = 0 \Rightarrow \frac{d}{dt}\gamma m c^2 = q \cdot \vec{E} \cdot \vec{v}$$

$$\mu = i \Rightarrow \frac{d}{dt}(\gamma m \vec{v}) = q(\vec{E} + \vec{\beta} \times \vec{B})$$

(b) $$P_i \equiv \frac{\partial L}{\partial v_i} = \gamma m v_i + \frac{q}{c} A_i \qquad (11)$$

In space-time co-ordinates,

$$H \equiv \vec{P} \cdot \vec{v} - L$$

$$= \left(\gamma m \vec{v} + \frac{q}{c}\vec{A}\right) \cdot \vec{v} + \frac{mc^2}{\gamma} + q\left(\Phi - \frac{\vec{v}\cdot\vec{A}}{c}\right)$$

$$= \gamma m c^2 \left(\frac{v^2}{c^2} + \frac{1}{\gamma^2}\right) + q\Phi$$

$$\therefore \quad H = \gamma m c^2 + q\Phi \qquad (12)$$

In covariant form, the Hamiltonian has a different meaning. Since $\tilde{H}$ contains $\tilde{L}$, it must be a Lorentz scalar, not an energy-like quantity. In terms of $\tilde{L}$, the covariant Hamiltonian may be defined by

$$\tilde{H} = p^\mu u_\mu - \tilde{L}$$

where $$p^\mu \equiv + \frac{\partial \tilde{L}}{\partial (u_\mu)} .$$

$$\therefore \quad p^\mu = - m u^\mu - \frac{q}{c} A^\mu$$

So $$\tilde{H} = - m u^\mu u_\mu - \frac{q}{c} u_\mu A^\mu$$
$$+ \frac{1}{2} m u_\mu u^\mu + \frac{q}{c} u_\mu A^\mu$$
$$= -\frac{1}{2} m u^\mu u_\mu$$
$$= -\frac{1}{2} m c^2$$

a Lorentz scalar.

<u>Note</u>: A better choice for $\tilde{L}$ would have been $\tilde{L} = \frac{1}{2} m u_\alpha u^\alpha + \frac{q}{c} u_\alpha A^\alpha$ (i.e. simply the negative of the given one), for then $\tilde{H} = +\frac{1}{2} mc^2$, which might have more intuitive appeal.

Problem 3 :

(a) Let $$L_2 = L_1 + \frac{d}{dt} f(q_i, t), \tag{1}$$

where $$L_1 = L_1(q_i(t), \dot{q}_i(t), t). \tag{2}$$

We know from Hamilton's principle ($\delta I = 0$) that

$$\frac{d}{dt}\left(\frac{\partial L_1}{\partial \dot{q}_i}\right) - \frac{\partial L_1}{\partial q_i} = 0. \tag{3}$$

The variational principle also implies

$$A \equiv \frac{d}{dt}\left(\frac{\partial L_2}{\partial \dot{q}_i}\right) - \frac{\partial L_2}{\partial q_i} = 0. \tag{4}$$

The question is, do (3) & (4) give the same equations of motion. From (4),

$$A = \frac{d}{dt}\left(\frac{\partial L_1}{\partial \dot{q}_i} + \frac{\partial}{\partial \dot{q}_i}\frac{d}{dt} f(q_i, t)\right) - \frac{\partial L_1}{\partial q_i}$$

$$- \frac{\partial}{\partial q_i}\frac{df}{dt} = 0$$

But $\frac{\partial}{\partial \dot{q}_i}\frac{d}{dt} f =$

$$\frac{\partial}{\partial \dot{q}_i}\left\{\frac{\partial f}{\partial t} + \frac{\partial f}{\partial q_j}\dot{q}_j\right\} = \frac{\partial f}{\partial q_i}.$$

Thus,

$$\frac{d}{dt}\left(\frac{\partial L_1}{\partial \dot{q}_i} + \frac{\partial f}{\partial q_i}\right) - \frac{\partial L_1}{\partial q_i} - \frac{\partial}{\partial q_i}\frac{df}{dt} = 0$$

Now, $$\frac{d}{dt}\left(\frac{\partial f}{\partial q_i}\right) - \frac{\partial}{\partial q_i}\frac{df}{dt}$$

$$= \frac{\partial^2 f}{\partial t\,\partial q_i} + \frac{\partial^2 f}{\partial q_j \partial q_i}\dot{q}_j - \frac{\partial^2 f}{\partial q_i \partial t} - \frac{\partial^2 f}{\partial q_i\,\partial q_j}\dot{q}_j$$

$$= 0 \ .$$

Thus, $$\frac{d}{dt}\left(\frac{\partial L_1}{\partial \dot{q}_i}\right) - \frac{\partial L_1}{\partial q_i} = 0 \ , \qquad (5)$$

which is exactly in agreement with (3).

(b) $$L_1 = -\frac{mc^2}{\gamma} - q\left(\Phi - \frac{\vec{v}\cdot\vec{A}}{c}\right)$$

$$= -\frac{mc^2}{\gamma} - \frac{q}{\gamma c}U_\alpha A^\alpha . \qquad (6)$$

Consider $$L_2 = L_1 - \frac{q}{\gamma c}U_\alpha \partial^\alpha \Lambda . \qquad (7)$$

Then, $$\frac{\partial L_2}{\partial v_i} = \frac{\partial L_1}{\partial v_i} + \frac{q}{c}\frac{\partial \Lambda}{\partial x_i} \qquad (8)$$

and $$\frac{\partial L_2}{\partial x_i} = \frac{\partial L_1}{\partial x_i} - \frac{q}{c}\left(-\frac{\partial}{\partial x_i}\frac{\partial \Lambda}{\partial t} - \frac{\partial}{\partial x_i}\vec{v}\cdot\vec{\nabla}\Lambda\right)$$

But $$\frac{d}{dt}\left(\frac{\partial \Lambda}{\partial x_i}\right) = \frac{\partial^2 \Lambda}{\partial t\,\partial x_i} + \vec{v}\cdot\vec{\nabla}\cdot\frac{\partial \Lambda}{\partial x_i}.$$

Thus L_2 results in the same Euler-Lagrange equations as L_1.

Problem 4:

(a)
$$\mathcal{L} = -\frac{1}{8\pi}\partial_\alpha A_\beta \partial^\alpha A^\beta - \frac{1}{c} J_\alpha A^\alpha . \qquad (1)$$

The Euler-Lagrange equations are

$$\partial^\mu \frac{\partial \mathcal{L}}{\partial(\partial^\mu A^\nu)} = \frac{\partial \mathcal{L}}{\partial A^\nu} . \qquad (2)$$

Now,
$$\frac{\partial \mathcal{L}}{\partial(\partial^\mu A^\nu)} = -\frac{1}{4\pi}\partial_\mu A_\nu \qquad (3)$$

and
$$\frac{\partial \mathcal{L}}{\partial A^\nu} = -\frac{1}{c} J_\nu . \qquad (4)$$

Thus,
$$\partial^\mu \left\{ -\frac{1}{4\pi}\partial_\mu A_\nu \right\} = -\frac{1}{c} J_\nu$$

$$\therefore \quad \partial^\mu \partial_\mu A_\nu = \frac{4\pi}{c} J_\nu . \qquad (5)$$

This is just the wave equation for A_ν in the Lorentz gauge. It is trivial to show that these are in fact just the Maxwell equations:

$$\partial^\mu \partial_\mu A^\alpha = \partial_\mu \partial^\mu A^\alpha - \partial^\alpha (\partial_\mu A^\mu)$$

(since $\partial_\mu A^\mu = 0$ in the Lorentz gauge).

Thus,
$$\partial^\mu(\partial_\mu A^\alpha) = \partial_\mu \{ \partial^\mu A^\alpha - \partial^\alpha A^\mu \}$$
$$= \partial_\mu F^{\mu\alpha}$$

$$\therefore \quad (5) \Rightarrow \quad \partial_\mu F^{\mu\alpha} = \frac{4\pi}{c} J^\alpha \qquad (6)$$

which is the inhomogeneous Maxwell equation.

The homogeneous equations are satisfied automatically because of the definition of $F^{\alpha\beta}$ (see Jackson p. 597).

(b)
$$\frac{1}{16\pi} F_{\alpha\beta} F^{\alpha\beta} = \frac{1}{16\pi} (\partial_\alpha A_\beta - \partial_\beta A_\alpha)(\partial^\alpha A^\beta - \partial^\beta A^\alpha)$$

$$= \frac{1}{16\pi} \{ \partial_\alpha A_\beta \partial^\alpha A^\beta + \partial_\beta A_\alpha \partial^\beta A^\alpha$$
$$- \partial_\beta A_\alpha \partial^\alpha A^\beta - \partial_\alpha A_\beta \partial^\beta A^\alpha \}$$

$$= \frac{1}{8\pi} \{ \partial_\alpha A_\beta \partial^\alpha A^\beta - \partial_\alpha A_\beta \partial^\beta A^\alpha \}$$

Thus, the difference is the term

$$B \equiv \frac{1}{8\pi} \partial_\alpha A_\beta \partial^\beta A^\alpha$$

$$= \frac{1}{8\pi} \partial_\alpha (A_\beta \partial^\beta A^\alpha) - \frac{1}{8\pi} A_\beta \partial_\alpha \partial^\beta A^\alpha$$

But $A_\beta \partial_\alpha \partial^\beta A^\alpha = A_\beta \partial^\beta \partial_\alpha A^\alpha$

$= 0$. because $\partial_\alpha A^\alpha = 0$ in the Lorentz gauge. Thus

$$B = \frac{1}{8\pi} \partial_\alpha V^\alpha$$

where $V^\alpha \equiv A_\beta \partial^\beta A^\alpha$.

To see the effect on the equations of motions. Consider

$$\frac{\partial B}{\partial(\partial^\mu A^\nu)} = \frac{\partial_\nu A_\mu}{8\pi}$$

$$\frac{\partial B}{\partial A^\nu} = 0.$$

Thus $\partial^\mu \left(\frac{\partial_\nu A_\mu}{8\pi}\right) = \frac{1}{8\pi} \partial_\nu \partial^\mu A_\mu = 0.$

Again. $\partial^\mu A_\mu = 0$ in the Lorentz gauge.

Thus this difference has no effect on the Euler - Lagrange equations of motions