Solutions to Homework # 4

Problem 1: We have

$$\vec{A}(\vec{x},t) = \frac{1}{c}\int d^3x' \int dt' \, \frac{\vec{J}(\vec{x}\,',t')}{|\vec{x}-\vec{x}\,'|} \, \delta\left(t'-t+\frac{|\vec{x}-\vec{x}\,'|}{c}\right). \quad (1)$$

Now let's put

$$\vec{J}_\omega(\vec{x}) = \int \vec{J}(\vec{x},t)\, e^{i\omega t}\, dt , \quad (2)$$

$$\vec{A}_\omega(\vec{x}) = \int \vec{A}(\vec{x},t)\, e^{i\omega t}\, dt . \quad (3)$$

Thus, taking the Fourier transform of (1), we get

$$\int \vec{A}(\vec{x},t)\, e^{i\omega t}\, dt = \frac{1}{c}\int d^3x' \int dt' \int dt \, e^{i\omega t} \, \frac{\vec{J}(\vec{x}\,',t')}{|\vec{x}-\vec{x}\,'|} \, \delta\left(t'-t+\frac{|\vec{x}-\vec{x}\,'|}{c}\right).$$

$$\therefore \quad \vec{A}_\omega(\vec{x}) = \frac{1}{c}\int d^3x' \int dt' \, e^{i\omega\left[t'+\frac{|\vec{x}-\vec{x}\,'|}{c}\right]} \, \frac{\vec{J}(\vec{x}\,',t')}{|\vec{x}-\vec{x}\,'|}$$

$$= \frac{1}{c}\int d^3x' \, \frac{\vec{J}_\omega(\vec{x}\,')}{|\vec{x}-\vec{x}\,'|} \, e^{ik|\vec{x}-\vec{x}\,'|} \quad (4)$$

where $k \equiv \omega/c$.

Let us choose an origin of coordinates inside the source of size L. Then, at field points such that $|\vec{x}| >> L$, we may make the approximation

$$|\vec{x}-\vec{x}'| \approx |\vec{x}| - \frac{\vec{x}\cdot\vec{x}'}{|\vec{x}|} . \tag{5}$$

Thus,

$$\vec{A}_\omega(\vec{x}) \approx \frac{e^{ik|\vec{x}|}}{c|\vec{x}|} \int \vec{J}_\omega(\vec{x}')\, e^{-ik\frac{\vec{x}\cdot\vec{x}'}{|\vec{x}|}}\, d^3x'. \tag{6}$$

Since $kL << 1$ here, we may expand the exponential, so that

$$\vec{A}_\omega(\vec{x}) \approx \frac{e^{ik|\vec{x}|}}{c|\vec{x}|} \sum_{n=0}^{\infty} \frac{1}{n!} \int \vec{J}_\omega(\vec{x}') \left(-ik\frac{\vec{x}\cdot\vec{x}'}{|\vec{x}|}\right)^n d^3x . \tag{7}$$

The factor $e^{ik|\vec{x}|}$ expresses the effect of retardation from the source as a whole. The factor $e^{-ik\frac{\vec{x}\cdot\vec{x}'}{|\vec{x}|}}$ expresses the relative retardation of each element of the source.

Problem 2: Let the charge move in the x-y plane, and its position be denoted by $\vec{x}_0(t)$.

Then,

$$\vec{x}_0(t) = r_0 (\cos \omega_0 t \, \hat{x} + \sin \omega_0 t \, \hat{y})$$

$$\dot{\vec{x}}_0(t) = \omega_0 r_0 (- \sin \omega_0 t \, \hat{x} + \cos \omega_0 t \, \hat{y}) . \qquad (1)$$

The current is

$$\vec{J}(\vec{x}, t) = q \, \dot{\vec{x}}_0(t) \, \delta(\vec{x} - \vec{x}_0(t)) . \qquad (2)$$

Since $\vec{J}(\vec{x}, t)$ is clearly periodic, we write it as a Fourier series

$$\vec{J}(\vec{x}, t) = \vec{a}_0(\vec{x}) + \sum_{n=1}^{\infty} \Big[\vec{a}_n(\vec{x}) \cos(n\omega_0 t)$$

$$+ \vec{b}_n(\vec{x}) \sin(n\omega_0 t) \Big] \qquad (3)$$

where

$$\vec{a}_n(\vec{x}) = \frac{1}{\pi} \int_0^{2\pi} \vec{J}(\vec{x}, t) \cos(n\omega_0 t) \, d(\omega_0 t) \qquad (4)$$

$$\vec{b}_n(\vec{x}) = \frac{1}{\pi} \int_0^{2\pi} \vec{J}(\vec{x}, t) \sin(n\omega_0 t) \, d(\omega_0 t) . \qquad (5)$$

Note that both sin & cos terms must be kept here because of the form of $\vec{x}_0$.

For this expansion of $\vec{J}(\vec{x},t)$, the corresponding expansion for $\vec{A}(\vec{x},t)$ is

$$\vec{A}(\vec{x},t) = \sum_{n=1}^{\infty}\left[\vec{A}_n^{a}(\vec{x})\cos(n\omega_0 t) + \vec{A}_n^{b}(\vec{x})\sin(n\omega_0 t)\right], \tag{6}$$

where $$\vec{A}_n^{a}(\vec{x}) = C_a(k,|\vec{x}|)\sum_{m=1}^{\infty}\frac{1}{m!}\int \vec{a}_n(\vec{x}')\left(\frac{\vec{x}\cdot\vec{x}'}{|\vec{x}|}\right)^{(m-1)} d^3x'$$

$$\vec{A}_n^{b}(\vec{x}) = C_b(k,|\vec{x}|)\sum_{m=1}^{\infty}\frac{1}{m!}\int \vec{b}_n(\vec{x}')\left(\frac{\vec{x}\cdot\vec{x}'}{|\vec{x}|}\right)^{(m-1)} d^3x' \tag{7}$$

If we now substitute the expressions for $\vec{a}_n$ and $\vec{b}_n$ into (7), and perform the d^3x' integral first, we get for the <u>dipole</u> contribution $(\ell=1)$

$$\left[\vec{A}_n^{a}(\vec{x})\right]_1 \propto \int_0^{2\pi}\left(-\sin\omega_0 t\ \hat{x} + \cos\omega_0 t\ \hat{y}\right)\cos(n\omega_0 t)\, d(\omega_0 t) \propto \delta_{n,1}\ \hat{y} \tag{8}$$

and $\quad [\vec{A}_n^{\,b}(\vec{x})]_1 \propto \delta_{n,1}\, \hat{x}$. $\qquad$ (9)

Thus, the <u>dipole</u> contribution to the vector potential is nonzero only at <u>$n=1$ $(\omega=\omega_0)$</u>.

Similarly, for the quadrupole contribution, we obtain

$$[\vec{A}_n^{\,a}(\vec{x})]_2 \propto \int_0^{2\pi} (-\sin\omega_0 t\, \hat{x} + \cos\omega_0 t\, \hat{y}) \cdot$$

$$\left(\frac{x}{|\vec{x}|}\cos\omega_0 t + \frac{y}{|\vec{x}|}\sin\omega_0 t\right)\cos(n\omega_0 t)\, d(\omega_0 t).$$

Using standard trigonometric identities, we can write this as

$$[\vec{A}_n^{\,a}(\vec{x})]_2 \propto \frac{\hat{x}}{2}\int_0^{2\pi}\left[-\frac{x}{|\vec{x}|}\sin(2\omega_0 t) - \frac{y}{|\vec{x}|}\left(1 - \cos(2\omega_0 t)\right)\right]\cdot \cos(n\omega_0 t)\, d(\omega_0 t)$$

$$+ \frac{\hat{y}}{2}\int_0^{2\pi}\left[\frac{x}{|\vec{x}|}\left(1+\cos(2\omega_0 t)\right) + \frac{y}{|\vec{x}|}\sin(2\omega_0 t)\right]\cdot \cos(n\omega_0 t)\, d(\omega_0 t)$$

$$\propto \frac{\hat{x}}{2}\frac{y}{|\vec{x}|}\delta_{n,2} + \frac{\hat{y}}{2}\frac{x}{|\vec{x}|}\delta_{n,2} \qquad (10)$$

Analogously, we have

$$\left[\vec{A}^{\,b}_{n}(\vec{x}) \right]_2 \propto \frac{\hat{x}}{2} \frac{x}{|\vec{x}|} \delta_{n,2} + \frac{\hat{y}}{2} \frac{y}{|\vec{x}|} \delta_{n,2} \, . \qquad (11)$$

Thus, the quadrupole contribution is nonzero only at $n=2$ $(\omega = 2\omega_0)$. In general, the l-pole contribution is solely at the harmonic $\omega = l\omega_0$.

Problem 3 :

Let the density at time t be $\rho(t)$. (We already know ρ is uniform for a given t). Then

$$\int \rho(t)\, d^3x = \rho(t) \int_0^{\pi} d\theta \int_0^{2\pi} d\phi \int_0^{R(\theta)} r^2 dr \sin\theta$$

$$= \rho(t)\, 2\pi \int_0^{\pi} d\theta \sin\theta\, R_0^3 \tfrac{1}{3}\left(1 + \beta \tfrac{1}{2}[3\cos^2\theta - 1]\right)$$

$$= \rho(t) \frac{2\pi R_0^3}{3} \left\{ 2 - \beta + \beta \right\}$$

$$= \frac{4\pi R_0^3}{3} \rho(t) \equiv Q$$

$$\therefore \quad \rho(t) = \frac{Q}{4\pi R_0^3/3} = \text{constant}$$

Thus, ρ is not only uniform in x, it is uniform in t as well.

In the long wavelength approximation,

$$a_E(l,m) \approx \frac{4\pi k^{l+2}}{i(2l+1)!!} \left(\frac{l+1}{l}\right)^{1/2} Q_{lm}, \qquad (1)$$

where $$Q_{lm} \equiv \int r^l\, Y^*_{lm}\, \rho\, d^3x, \qquad (2)$$

and $$a_M(l,m) = 0 \quad , \text{ since } \vec{r}\times\vec{J} = 0. \qquad (3)$$

Thus,

$$Q_{\ell m} = \int_0^{\pi} d\theta \cdot \sin\theta \int_0^{2\pi} d\phi \int_0^{R(\theta)} dr \cdot r^{\ell+2} \, \rho \, Y_{\ell m}^*$$

$$= \frac{1}{\ell+3} \frac{3Q}{4\pi R_0^3} \int_0^{\pi} d\theta \cdot \sin\theta \int_0^{2\pi} d\phi \; R_0^{\ell+3} \left[1+\beta P_2(\cos\theta)\right]^{\ell+3} Y_{\ell m}^*$$

$$\approx \frac{3QR_0^{\ell}}{(\ell+3)4\pi} \int_0^{\pi} d\theta \cdot \sin\theta \int_0^{2\pi} d\phi \left(1 + [\ell+3]\beta P_2(\cos\theta)\right) Y_{\ell m}^*$$

$$\approx \frac{3QR_0^{\ell}}{(\ell+3)4\pi} \int_0^{\pi} d\theta \sin\theta \int_0^{2\pi} d\phi \; Y_{\ell m}^* +$$

$$\frac{3QR_0^{\ell}}{4\pi} \beta \int_0^{\pi} d\theta \sin\theta \int_0^{2\pi} d\phi \sqrt{\frac{4\pi}{2\ell+1}} \, Y_{20} \, Y_{\ell m}^*$$

$$= \frac{3QR_0^{\ell}}{(\ell+3)4\pi} \int_0^{\pi} d\theta \sin\theta \int_0^{2\pi} d\phi \sqrt{4\pi} \, Y_{00} \, Y_{\ell m}^*$$

$$+ \frac{3QR_0^{\ell}}{4\pi} \beta \sqrt{\frac{4\pi}{2\ell+1}} \int_0^{\pi} d\theta \sin\theta \int_0^{2\pi} d\phi \, Y_{20} \, Y_{\ell m}^*$$

$$= \frac{3QR_0^{\ell}}{(\ell+3)\sqrt{4\pi}} \delta_{\ell 0} \delta_{m0} + \frac{3QR_0^{\ell}}{\sqrt{(2\ell+1)4\pi}} \beta \, \delta_{\ell 2} \delta_{m0}$$

$$\therefore \quad Q_{00} = \frac{Q}{\sqrt{4\pi}}$$

$$Q_{20} = \frac{3QR_0^2}{\sqrt{20\pi}} \beta$$

All others are zero.

In the radiation zone,

$$\vec{B} \rightarrow \frac{e^{ikr - i\omega t}}{kr} \sum_{lm} (-i)^{l+1} \left[a_E(l,m) \vec{X}_{lm} + a_M(l,m) \hat{n} \times \vec{X}_{lm} \right]$$

and $\vec{E} \rightarrow \vec{B} \times \hat{n}$, so that

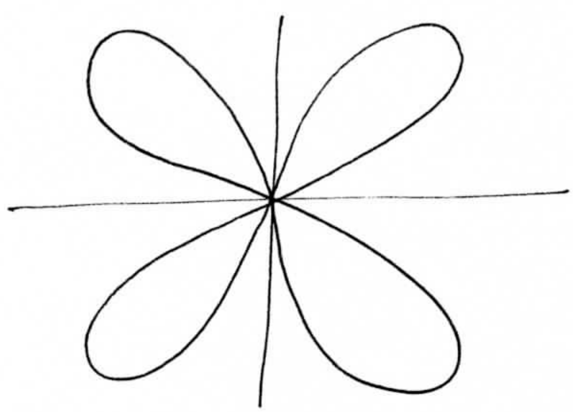
$$\frac{dP}{d\Omega} = \frac{c}{8\pi k^2} \left| \sum_{lm} (-i)^{l+1} \left[a_E(l,m) \vec{X}_{lm} \times \hat{n} + a_M(lm) \vec{X}_{lm} \right] \right|^2$$

Only Q_{20} contributes to the radiation since Q_{00} is t-independent. Thus,

$$\frac{dP}{d\Omega} = \frac{dP(2,0)}{d\Omega} = | a_E(2,0) |^2 | \vec{X}_{20} |^2 \frac{c}{8\pi k^2}$$

$$= \frac{c}{8\pi k^2} \left(\frac{15}{8\pi} \right) \sin^2\theta \cos^2\theta \left\{ \frac{4\pi k^4}{5!!} \left(\frac{3}{2} \right)^{1/2} \frac{3 Q R_0^2}{\sqrt{20\pi}} \beta \right\}^2$$

$$= \frac{9c}{800\pi} k^6 Q^2 R_0^4 |\beta|^2 \sin^2\theta \cos^2\theta \qquad (4)$$



("standard quadrupole")

The total radiated power is

$$P(2,0) = \frac{c}{8\pi k^2} \left| a_E(2,0) \right|^2$$

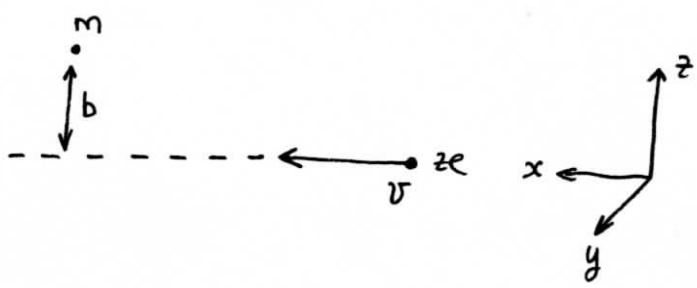
$$= \frac{c}{8\pi k^2} \left| \frac{4\pi k^4}{5!!} \left(\frac{3}{2}\right)^{1/2} \frac{3 Q R_0^2}{\sqrt{20\pi}} \beta \right|^2$$

$$= \frac{3c}{500} k^6 Q^2 R_0^4 |\beta|^2 .$$

Problem 4 :

(a) In the rest frame of the incident particle,

$$\Phi'(r') = ze \frac{e^{-k_D r'}}{r'} \tag{1}$$



Thus,

$$E_x' = -\frac{\partial \Phi'(r')}{\partial x'} = \frac{ze\,(1+k_D r')}{r'^3}\, e^{-k_D r'}\, x'$$

$$E_y' = \frac{ze\,(1+k_D r')}{r'^3}\, e^{-k_D r'}\, y'$$

$$E_z' = \frac{ze\,(1+k_D r')}{r'^3}\, e^{-k_D r'}\, z'$$

$$B_x' = B_y' = B_z' = 0 .$$

In the electrons rest frame, we now have

$$E_x = E_x' = \frac{ze\,(1 + k_D r')}{r'^3}\, e^{-k_D r'}\, x'$$

$$E_y = \gamma_v\, E_y' \simeq E_y' = \frac{ze\,(1 + k_D r')}{r'^3}\, e^{-k_D r'}\, y'$$

$$E_z = \gamma_v\, E_z' \approx E_z' = \frac{ze\,(1 + k_D r')}{r'^3}\, e^{-k_D r'}\, z' .$$

But
$$x' \simeq vt$$
$$y' = 0$$
$$z' = b,$$

so that

$$E_x \propto t$$
$$E_y = 0$$
$$E_z = \frac{ze\left(1 + k_D\sqrt{b^2 + (vt)^2}\right)}{\left[\,b^2 + (vt)^2\,\right]^{3/2}}\, e^{-k_D\sqrt{b^2+(vt)^2}} \cdot b$$

Since E_x is anti-symmetric in t, Δp_x is still 0.

Only the transverse component survives:

$$\Delta p = \int_{-\infty}^{\infty} e\, E_z(t)\, dt$$

$$= ze^2 k_D^2 \int_{-\infty}^{\infty} dt \; (k_D b) \; \frac{\left(1 + \sqrt{(k_D b)^2 + (k_D v t)^2}\right)}{\left[(k_D b)^2 + (k_D v t)^2\right]^{3/2}} \; e^{-\sqrt{(k_D b)^2 + (k_D v t)^2}}$$

$$\therefore \quad \frac{d^2}{d(k_D b)^2} (\Delta p) + \frac{1}{(k_D b)} \frac{d}{d(k_D b)} (\Delta p) - \left(1 + \frac{1}{(k_D b)^2}\right) \Delta p = 0$$

The solution to this equation is the modified Bessel function $K_1(k_D b)$:

$$(\Delta p) \propto K_1(k_D b).$$

Since $\quad \Delta p\,(k_D \to 0) = \dfrac{2ze^2}{bv} = \dfrac{2ze^2}{k_D b v} \cdot k_D$,

and $\quad K_1(k_D b) \xrightarrow{k_D b \ll 1} \dfrac{\Gamma(1)}{2} \left(\dfrac{2}{k_D b}\right)$, we

need $\quad \boxed{\Delta p = \dfrac{2\, ze^2 k_D}{v} K_1(k_D b)}$.

Thus, $\quad \Delta E(b) \simeq \dfrac{(\Delta p)^2}{2m} = \dfrac{2\,(ze^2)^2}{mv^2} k_D^2 K_1^2(k_D b).$

(b) For $k_D b < 1$,

$$K_1(k_D b) \rightarrow \frac{1}{k_D b}$$

$$\therefore \quad \Delta E \simeq \frac{2(ze^2)^2}{M v^2 b^2}$$

In a distance dx, the particle loses an amount of energy

$$dE = dx \int_{b_{min}}^{1/k_D} 2\pi b\, db\, n_0\, \Delta E$$

$$\therefore \quad \frac{dE}{dx} = 2\pi n_0 \int_{b_{min}}^{1/k_D} \frac{2(ze^2)^2}{M v^2 b^2}\, b\, db$$

$$= \frac{4\pi n_0 e^2}{m} \frac{(ze)^2}{v^2} \int_{b_{min}}^{1/k_D} \frac{db}{b}$$

$$= \frac{(ze)^2}{v^2} \omega_p^2 \left[\ln(1/k_D) - \ln(b_{min}) \right]$$

$$= \frac{(ze)^2}{v^2} \omega_p^2 \ln\left(\frac{1}{k_D b_{min}}\right)$$