Solutions to Homework 5

Problem 1: In class, we found that the potential from a point charge could be written

$$A^\mu(x) = 2q \int d^4x' \, \theta(x_0 - x_0') \, \delta[(x-x')^2] \int d\tau \, U^\mu(\tau) \, \delta^{(4)}(x' - x_q(\tau)) \, . \qquad (1)$$

Let's write this as

$$A^\mu(x) = 2q \int d\tau \int d^4x' \, \theta(x_0 - x_0') \, \delta[(x-x')^2] \, U^\mu(\tau) \, \delta^{(4)}(x' - x_q(\tau)) . \qquad (2)$$

Then, integrating over d^4x', we get

$$A^\mu(x) = 2q \int d\tau \, \theta(x_0 - x_q^0(\tau)) \, \delta[(x - x_q(\tau))^2] \, U^\mu(\tau) . \qquad (3)$$

To proceed further, we need to use the identity

$$\int d\tau \, f(\tau) \, \delta(g(\tau)) = \sum_i \frac{f(\tau_i)}{\left| \frac{dg}{d\tau} \right|_{\tau=\tau_i}} , \qquad (4)$$

where $g(\tau_i) = 0$.

Put $\quad g(\tau) = (x - x_q(\tau))^2$, with

$$(x - x_q(\tau))^2 = (x_0 - x_q^0(\tau))^2 - (\vec{x} - \vec{x}_q(\tau))^2 \qquad (5)$$
$$= [\, x_0 - x_q^0(\tau) + |\vec{x} - \vec{x}_q(\tau)|\,] \cdot [\, x_0 - x_q^0(\tau) - |\vec{x} - \vec{x}_q(\tau)|\,]$$

The zeros of $g(\tau)$ are given by

Light cone conditions
$$\begin{cases} x_0 - x_q^0(\tau_0^+) = |\vec{x} - \vec{x}_q(\tau_0^+)|, & (6) \\ x_0 - x_q^0(\tau_0^-) = -|\vec{x} - \vec{x}_q(\tau_0^-)|. & (7) \end{cases}$$

Then,
$$\frac{dg}{d\tau} = 2\,(x^\mu - x_q^\mu(\tau))\, U_\mu(\tau)$$

$$\therefore \quad \left|\frac{dg}{d\tau}\right|_{\tau_0} = 2\,(x^\mu - x_q^\mu(\tau_0))\, U_\mu(\tau_0) \qquad (8)$$

Thus,
$$A^\mu(x) = \left.\frac{q\, U^\mu(\tau)}{(x^\alpha - x_q^\alpha(\tau))\, U_\alpha(\tau)}\right|_{\tau = \tau_0^+} \qquad (9).$$

Note that although 2 solutions τ_0^+, τ_0^- are possible, only one (the retarded one) is allowed by the θ-function in the Green's function. As you would expect, the charge contributes to the potential $A^\mu(x)$ only at the retarded time τ_0^+ defined by the light-cone condition

$$(x - x_q(\tau_0^+))^2 = 0 . \qquad (10)$$

We can express these Liénard-Wiechert potentials in non-covariant form by writing

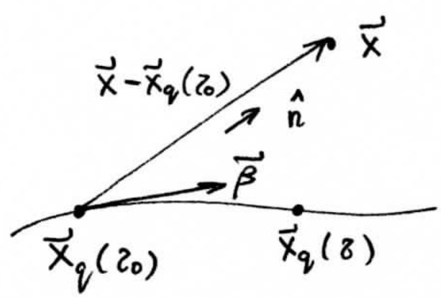
$$U_\alpha \left(x^\alpha - x^\alpha_q(\tau_0)\right) =$$

$$U_0 \left(x_0 - x^0_q(\tau_0)\right) - \vec{U}\cdot\left(\vec{x} - \vec{x}_q(\tau_0)\right)$$

$$= \gamma c \,|\vec{x} - \vec{x}_q(\tau_0)| - \gamma\vec{v}\cdot\left(\vec{x} - \vec{x}_q(\tau_0)\right)$$

(by using (10))

$$= \gamma c\, |\vec{x} - \vec{x}_q(\tau_0)| \left(1 - \vec{\beta}\cdot\hat{n}\right)$$



Thus, using $A^{\mu} = (\Phi, \vec{A})$, (9) becomes

$$\Phi(\vec{x},t) = \left[\frac{q}{(1-\vec{\beta}\cdot\hat{n})\,|\vec{x}-\vec{x}_q|} \right]_{ret} , \tag{11}$$

$$\vec{A}(\vec{x},t) = \left[\frac{q\vec{\beta}}{(1-\vec{\beta}\cdot\hat{n})\,|\vec{x}-\vec{x}_q|} \right]_{ret} , \tag{12}$$

where "ret" means the quantities are to be evaluated at the retarded time τ_0 :

$$x_0 - x_q^0(\tau_0) = |\vec{x} - \vec{x}_q(\tau_0)| . \tag{13}$$

Problem 2: (a) In the frame of reference K' instantaneously at rest with respect to the particle, the particle emits an amount of energy dw' in time dt'. Since the momentum of this radiation is zero $(d\vec{p}' = \vec{0})$, due to the symmetric emission in this frame, the energy in a frame K moving with velocity $-v$ w.r.t. the particle is

$$dW = \gamma dW'. \qquad (1)$$

The time interval dt is simply

$$dt = \gamma dt'. \qquad (2)$$

The power emitted is thus

$$P = \frac{dW}{dt} \quad \text{in } K \qquad (3)$$

$$P' = \frac{dW'}{dt'} \quad \text{in } K', \qquad (4)$$

where

$$P = P'. \qquad (5)$$

Thus, the total emitted power is a Lorentz invariant for any emitter that emits with front-back symmetry in its instantaneous rest frame.

(b) From the Larmor formula, we have

$$P' = \frac{2q^2}{3c^3} |\vec{a}'|^2 . \qquad (6)$$

Defining $a^\mu \equiv \frac{dU^\mu}{d\tau}$, we have

$$\begin{aligned} a^\mu \cdot U_\mu &= \frac{dU^\mu}{d\tau} U_\mu \\ &= \frac{1}{2}\frac{d}{d\tau}(U^\mu U_\mu) \\ &= \frac{1}{2}\frac{d}{d\tau}(c^2) \\ &= 0 \end{aligned} \qquad (7)$$

In the frame K', $U^\mu = (c, \vec{0})$, $a'_0 = 0$.

Thus, $|\vec{a}'|^2 = a'_k a'^k = a'_0 a'^0 + a'_k a'^k$

$$= a'^\alpha a'_\alpha . \qquad (8)$$

Thus, in covariant form,

$$\boxed{P = \frac{2q^2}{3c^3} a^\alpha a_\alpha} \qquad (9)$$

(c)

$$dt = \gamma\left(dt' + \frac{v}{c^2}dx'\right) \equiv \gamma\sigma\, dt'$$

$$\sigma' \equiv 1 + \frac{v}{c^2}\frac{dx'}{dt'} = 1 + \frac{v u_x'}{c^2}$$

$$du_x = \gamma^{-2}\sigma^{-2}\, du_x'$$

$$du_y = \gamma^{-1}\sigma^{-2}\left(\sigma\, du_y' - \frac{v u_y'}{c^2} du_x'\right)$$

$$\therefore \quad a_x = \frac{du_x}{dt} = \gamma^{-3}\sigma^{-3}\frac{du_x'}{dt'} = \gamma^{-3}\sigma^{-3} a_x'$$

$$a_y = \frac{du_y}{dt} = \gamma^{-2}\sigma^{-3}\left(\sigma\frac{du_y'}{dt'} - \frac{v u_y'}{c^2}\frac{du_x'}{dt'}\right)$$

$$= \gamma^{-2}\sigma^{-3}\left(\sigma a_y' - \frac{v u_y'}{c^2} a_x'\right)$$

$$a_z = \gamma^{-2}\sigma^{-3}\left(\sigma a_z' - \frac{v u_z'}{c^2} a_x'\right)$$

In K', $u_x' = u_y' = u_z' = 0$. Thus, $\sigma = 1$ and

$$a_\parallel' = \gamma^3 a_\parallel$$

$$a_\perp' = \gamma^2 a_\perp$$

Thus,
$$P = \frac{2q^2}{3c^3}\vec{a}'\cdot\vec{a}' = \frac{2q^2}{3c^3}\left(a_\parallel'^2 + a_\perp'^2\right)$$

$$= \frac{2q^2}{3c^3}\gamma^4\left(a_\perp^2 + \gamma^2 a_\parallel^2\right)$$

Problem 3 :

(a)

$$\frac{d}{dt}(\gamma m \vec{v}) = \frac{q}{c}\vec{v}\times\vec{B} \qquad (1)$$

$$\frac{d}{dt}(\gamma m c^2) = q\vec{v}\cdot\vec{E} = 0 \qquad (2)$$

$\therefore \quad \gamma =$ constant

$$\therefore \quad m\gamma \frac{d\vec{v}}{dt} = \frac{q}{c}\vec{v}\times\vec{B}$$

$$\therefore \quad \frac{d\vec{v}_{\parallel}}{dt} = 0$$

$$\frac{d\vec{v}_{\perp}}{dt} = \frac{q}{\gamma m c}\vec{v}_{\perp}\times\vec{B} \qquad (3)$$

where $\vec{v}_{\parallel}$ is parallel to $\vec{B}$ and $\vec{v}_{\perp}$ is perpendicular

The frequency of gyration (see previous problem sets) is thus

$$\omega_B = \frac{qB}{\gamma m c}, \qquad (4)$$

and the acceleration is

$$a_{\perp} = \omega_B v_{\perp}. \qquad (5)$$

Thus,

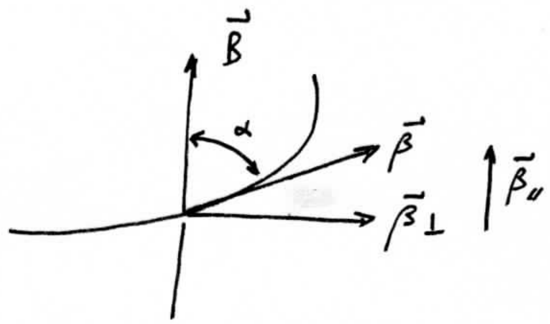
$$P = \frac{2q^2}{3c^3}\gamma^4 \frac{q^2B^2}{\gamma^2 m^2 c^2} v_{\perp}^2 . \qquad (6)$$

$$= \frac{2}{3} r_0^2 c\beta_{\perp}^2 \gamma^2 B^2$$

where $r_0 \equiv e^2/mc^2$.

It is necessary to average this over all angles for a given speed β. Let α be the pitch angle, which is the angle between field and velocity. Then

$$\langle \beta_\perp^2 \rangle = \frac{\beta^2}{4\pi} \int \sin^2\alpha \, d\Omega$$



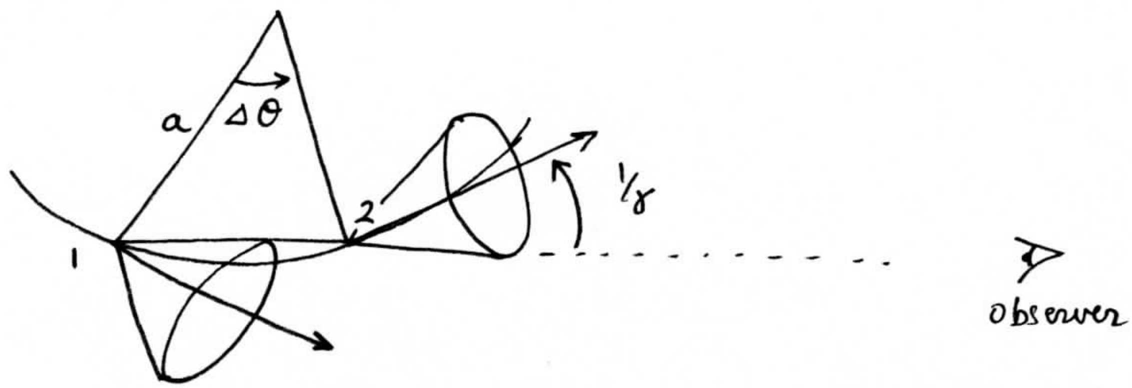
$$\therefore \quad \langle \beta_\perp^2 \rangle = \frac{2\beta^2}{3} \quad \text{so}$$

$$P = \left(\tfrac{2}{3}\right)^2 r_0^2 c \, \beta^2 \gamma^2 B^2 . \tag{7}$$

Putting $U_B = \dfrac{B^2}{8\pi}$, and $\sigma_T = \dfrac{8\pi r_0^2}{3}$,

$$\boxed{P = \frac{4}{3} \sigma_T c \, \beta^2 \gamma^2 U_B} \tag{8}$$

(b) In this simple picture, the observer will see the pulse from points 1 and 2 along the particle's path:



From the geometry, $\Delta\theta = 2/\gamma$, so that the arc length around the path is

$$\Delta s = \frac{2}{\gamma} a \quad . \tag{9}$$

We also know

$$\gamma m \frac{\Delta \vec{v}}{\Delta t} = \frac{q}{c} \vec{v} \times \vec{B}, \tag{10}$$

where $|\Delta \vec{v}| = v\,\Delta\theta$, $\Delta s = v\,\Delta t$. Thus,

$$\frac{\Delta\theta}{\Delta s} = \frac{|\Delta \vec{v}|}{v} \frac{1}{v\,\Delta t} = \frac{1}{v^2} \frac{q}{\gamma m c} |\vec{v} \times \vec{B}|$$

$$= \frac{q B \sin\alpha}{\gamma m c v}$$

$$\therefore \quad \Delta s = \frac{2}{\gamma} \cdot \frac{\gamma m c v}{q B \sin\alpha} = \frac{2v}{\gamma \omega_B \sin\alpha} \tag{11}$$

The times t_1 and t_2 at which the particle passes points 1 and 2 are such that

$$\Delta S = v(t_2 - t_1)$$

$$\therefore \quad t_2 - t_1 = \frac{2}{\gamma \omega_B \sin\alpha} . \qquad (12)$$

Thus the pulse duration is

$$\Delta t \equiv (t_2 - t_1)(1 - \vec{\beta} \cdot \hat{n}) = \frac{2}{\gamma \omega_B \sin\alpha} \left(1 - \frac{v}{c}\right) . \qquad (13)$$

Note that this takes into account the light travel time effect due to a difference ΔS in the path from points 1 and 2.

Problem 4

$$P = \frac{2q^2}{3c^3}\gamma^4\left(a_\perp^2 + \gamma^2 a_\parallel^2\right)$$

$$a_\parallel = \frac{1}{\gamma^3} a'_\parallel = \frac{1}{\gamma^3}\left(\frac{\vec{v}\cdot\vec{F}}{v}\right)\frac{1}{m}$$

$$\vec{a}_\perp = \frac{1}{\gamma^2}\vec{a}'_\perp = \frac{1}{\gamma^2}\left[\vec{F} - \frac{\vec{v}\cdot\vec{F}}{v^2}\vec{v}\right]\frac{1}{m}$$

$$\gamma^2 a_\parallel^2 = \frac{1}{\gamma^4}\frac{F_\parallel^2}{m^2}$$

$$\vec{a}_\perp^{\,2} = \frac{1}{\gamma^4}\frac{1}{m^2}\left[F^2 + F_\parallel^2 - 2F_\parallel^2\right]$$

$$= \frac{1}{\gamma^4}\frac{F^2 - F_\parallel^2}{m^2}$$

$$= \frac{1}{\gamma^4}\frac{F_\perp^2}{m^2}$$

So $$P = \frac{2q^2}{3c^3}\gamma^4\left[\frac{1}{\gamma^4}\frac{F_\perp^2}{m^2} + \frac{1}{\gamma^4}\frac{F_\parallel^2}{m^2}\right]$$

$$= \frac{2q^2}{3c^3}\frac{F^2}{m^2}$$

Remember that P is invariant, so $P = P'$, and P' is given by the Larmor formula applied to the particle in its own rest frame, where $\vec{a}' = \frac{\vec{F}}{m}$ in terms of the Newtonian force in that frame.

As seen in the lab, however,

$$\frac{dp^\mu}{d\tau} = f^\mu$$

so $$\frac{dp^\mu}{d\tau}\frac{dp_\mu}{d\tau} = (f^0)^2 - \vec{f}\cdot\vec{f}$$

where $$f^0 = \frac{\gamma}{c}\vec{v}\cdot\vec{F}$$

$$\vec{f} = \vec{F} + (\gamma-1)\vec{v}\,\frac{\vec{v}\cdot\vec{F}}{v^2}$$

Now, $$\vec{f}_\parallel = \vec{v}\,\frac{\vec{v}\cdot\vec{f}}{v^2} = \vec{v}\,\frac{\vec{v}\cdot\vec{F}}{v^2} + (\gamma-1)\vec{v}\,\frac{\vec{v}\cdot\vec{F}}{v^2}$$

$$= \gamma\vec{v}\,\frac{F_\parallel}{v}$$

$$\vec{f}_\perp = \vec{f} - \vec{f}_\parallel = \vec{F} + (\gamma-1)\vec{v}\,\frac{\vec{v}\cdot\vec{F}}{v^2} - \gamma\vec{v}\,\frac{F_\parallel}{v}$$

$$= \vec{F} - \frac{\vec{v}}{v}F_\parallel$$

$$= \vec{F}_\perp$$

So $$\left.\begin{aligned} \vec{F}_\parallel &= \frac{1}{\gamma}\vec{f}_\parallel \\ \vec{F}_\perp &= \vec{f}_\perp \end{aligned}\right\}$$

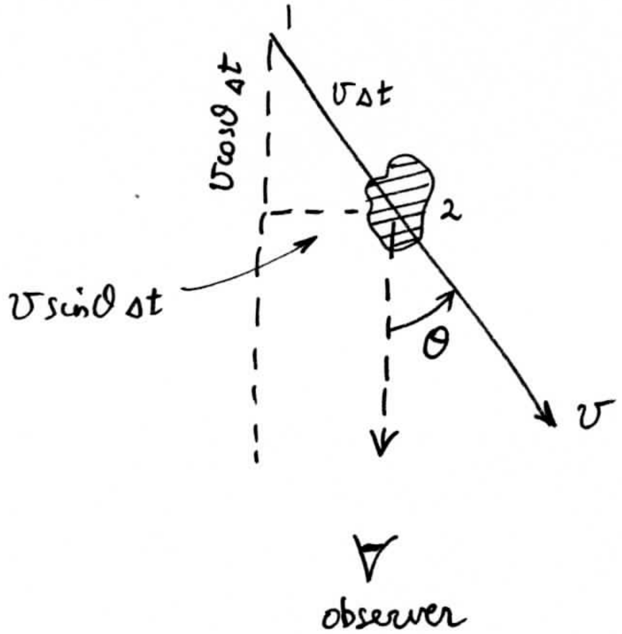
$$(f^0)^2 - \vec{f}\cdot\vec{f} = \gamma^2\beta^2F_\parallel^2 - \vec{f}_\parallel^{\,2} - \vec{f}_\perp^{\,2}$$
$$= -\frac{1}{\gamma^2}f_\parallel^2 - f_\perp^2$$

and therefore

$$P = \frac{2q^2}{3c^3}\frac{1}{m^2}\left(\frac{1}{\gamma^2}f_\parallel^2 + f_\perp^2\right)$$

$$\boxed{P = \frac{2q^2}{3c^3m^2\gamma^2}\left(f_\parallel^2 + \gamma^2 f_\perp^2\right)}$$

Problem 5:



(a) Suppose the blob moves from points 1 to 2 in a time Δt. Because 2 is closer to the observer than 1, the apparent time difference between light received by him/her $(\Delta t)_{app}$ is

$$(\Delta t)_{app} = \Delta t \left(1 - \frac{v}{c}\cos\theta\right) \qquad (1)$$

The apparent transverse velocity is thus

$$v_{app} = \frac{v \Delta t \sin\theta}{(\Delta t)_{app}}$$

$$= \frac{v \sin\theta}{1 - \frac{v}{c}\cos\theta} \qquad (2)$$

(b) Differentiation with respect to θ and setting to zero yields the critical angle θ_c :

$$\cos\theta_c = \frac{v}{c} \equiv \beta$$

$$\sin\theta_c = \sqrt{1-\beta^2} = \gamma^{-1} . \qquad (3)$$

Thus, the maximum apparent velocity is

$$v_{max} = \frac{v\sqrt{1-\beta^2}}{1-\beta^2} = \gamma v . \qquad (4)$$

This clearly exceeds c when $\gamma >> 1$.