PHYSICS 515-A

Standard Problems

Spring 2021

These are not to be handed in, but try to complete them before the end of the semester

[1] **Electrostatics** A box with sides of length a (along $\hat{x}$), b (along $\hat{y}$), and c (along $\hat{z}$), has potential $\Phi(x,y,z) = 0$ on all sides except the side at $z = c$, where Φ is a constant ($\Phi = V$). Find the electric potential everywhere inside the box.

Answer: $\Phi(x,y,z) = \sum_{n,m=1}^{\infty} A_{nm}\, sin(\alpha_n x) \sin(\beta_m y) \sinh(\gamma_{nm} z)$, where $A_{nm} = 4[ab \sinh(\gamma_{nm} c)]^{-1} \int_0^a dx \int_0^b dy\, V\, \sin(n\pi x/a)\, \sin(m\pi y/b)$.

[2] **Electrostatics** A hollow conducting sphere (radius a) is divided into equal parts by a thin insulating ring, and the two halves are maintained at potentials V_0 and 0.

(a) What is the Green function?

(b) Derive an integral expression for the potential *outside* the sphere.

(c) Show that along the z-axis, the potential is

$$V(r, \theta = 0) = \frac{V_0}{2r}\left[r + a - \frac{(r^2 - a^2)}{(r^2 + a^2)^{1/2}}\right] \quad .$$

[3] **Magnetostatics** A cylindrical conductor of radius a has a hole of radius b bored parallel to, and centered a distance d from, the cylinder axis ($d + b < a$). The current density, J, is uniform throughout the remaining metal of the cylinder, and parallel to the cylinder's axis. Find the magnitude and direction of the magnetic induction $\mathbf{B}$ anywhere inside the hole.

Answer: $\mathbf{B} = (2\pi/c) J d \hat{y}$, where $\hat{y}$ is an axis perpendicular to the axis of the cylinder and perpendicular to the line connecting the center of the cylinder and the center of the hole.

[4] **Magnetostatics** Consider a thin spherical shell of dielectric which has a radius R and rotates with an angular velocity ω about its z-axis. A constant surface charge of density σ is placed uniformly on the sphere, producing a uniform magnetic field that is proportional to σ and ω.

(a) Find the magnetic dipole moment and the magnetization of the sphere.

(b) Find the magnetic field, and magnetic induction, inside and outside the sphere.

(c) A constant torque N is applied parallel to ω. How long does it take for the shell to stop rotating?

Answer: time $= 16\pi^2\sigma^2\omega R^5/N$.

[5] **Waves** Consider electromagnetic waves in free space of the form

$$\mathbf{E}(x, y, z, t) = \mathbf{E_0}(x, y)\, e^{i(kz-\omega t)} \ ,$$

$$\mathbf{B}(x, y, z, t) = \mathbf{B_0}(x, y)\, e^{i(kz-\omega t)} \ ,$$

where $\mathbf{E_0}$ and $\mathbf{B_0}$ are in the xy-plane.

(a) Find the relation between k and ω, as well as the relation between $\mathbf{E_0}$ and $\mathbf{B_0}$. Show that $\mathbf{E_0}$ and $\mathbf{B_0}$ satisfy the relations for electrostatics and magnetostatics in free space.

Answer: $k = \omega/c$, $\mathbf{E_0}$, $\mathbf{B_0}$, and $\hat{z}$ are mutually perpendicular, and $\vec{\nabla} \cdot \mathbf{E_0}$, $\vec{\nabla} \cdot \mathbf{B_0}$.

(b) What are the boundary conditions for $\mathbf{E}$ and $\mathbf{B}$ on the surface of a perfect conductor?

Answer: $\hat{n} \times \mathbf{E} = 0$, $\hat{n} \cdot \mathbf{D} = 0$, $\hat{n} \times \mathbf{H} = \vec{K}$, and $\hat{n} \cdot \mathbf{B} = 0$, where $\vec{K}$ is the linear current per unit width on the surface.

(c) Consider a wave of this type propagating along a coaxial transmission line. Assume the entral cylinder and the outher sheath are perfect conductors. Sketch the electromagnetic field lines for a particular cross section. Indicate the signs of the charges and the directions of the currents in the conductors.

Answer: For any given cross section, $\mathbf{E_0}$ is radial because of the symmetry. Similarly, $\mathbf{B_0}$ is circular. The charges are positive on the inner conductor and negative on the outer. The current is along $+\hat{z}$ on the inner conductor and along $-\hat{z}$ on the outer.

(d) Derive expressions for $\mathbf{E}$ and $\mathbf{B}$ in the transmission line in terms of the charge per unit length λ and the current I in the central conductor.

Answer: $\mathbf{E} = \lambda\hat{r}/r$, and $\mathbf{B} = I\hat{\theta}/r$.

[6] **Reflection and Transmission** A plane-polarized electromagnetic wave of frequency ω and amplitude E_i is incident at an angle θ_i to the normal of the interface between two simple (linear) dielectrics of refractive indices n_1 and n_2. The incident electric field lies in the $x - z$ plane.

(a) Write down appropriate expressions for the incident, reflected, and transmitted electric and magnetic fields in terms of the propagation vectors $\mathbf{k_i}$, $\mathbf{k_r}$, and $\mathbf{k_t}$.

(b) Write down the appropriate boundary conditions for the tangential componets of the electric and magnetic fields at the interface of the dielectric.

(c) Explain why the frequencies of all the waves must be the same.

(d) Show that the wave vectors $\mathbf{k_i}$, $\mathbf{k_r}$, and $\mathbf{k_t}$ must lie in the same $x - z$ plane.

(e) Prove that $\theta_i = \theta_r$ and that $n_1 \sin\theta_i = n_2 \sin\theta_t$.

(f) At what incident angle θ_i does the amplitude of the reflected wave vanish?

[7] **Radiation** This one is a little long for a final exam question, but it has many of the elements you would need in such a problem if it were to appear on the final. An infinitesimally small electric dipole $p = p_0 e^{-i\omega t}$ is placed at the origin of a spherical co-ordinate system (r, θ, ϕ). Given

$$\mathbf{A}(r) = -i\omega\, p\, \frac{e^{ikr}}{r}\, \hat{z} \ ,$$

(a) calculate the magnetic and electric field components.

(b) Show that the Poynting vector is given by

$$\mathbf{S} = \frac{k^4 c p_0^2 \sin^2\theta}{r^2} \ .$$

(c) Show that the radiative power is given by

$$P = (8\pi/3) k^4 c p_0^2 \ .$$

[8] **Radiation** Consider an oscillating linear electric quadrupole, composed of a charge $-2Q$ at $z = 0$, a charge $+Q$ at $z = s$, and a charge $+Q$ at $z = -s$. Each charge is oscillating according to $Q(t) = Q_0 e^{-i\omega t}$.

(a) Calculate the electric and magnetic field intensities of the quadrupole in the radiation field ($r >> \lambda >> s$).

Answer:

$$\mathbf{E}(\mathbf{x}, t) = -i\frac{Qs^2}{r}\left(\frac{\omega}{c}\right)^3 \cos\theta \sin\theta e^{i\omega(t-r/c)}\, \hat{\theta} \ ,$$

$$\mathbf{B}(\mathbf{x}, t) = -i\frac{Qs^2}{r}\left(\frac{\omega}{c}\right)^3 \cos\theta \sin\theta e^{i\omega(t-r/c)}\, \hat{\phi} \ .$$

(b) Calculate the time-averaged Poynting vector in the same limit.

Answer:

$$\langle \mathbf{S}(\mathbf{x}) \rangle = \frac{c}{8\pi} \frac{Q^2 s^4}{r^2} \left(\frac{\omega}{c}\right)^6 \cos^2\theta \sin^2\theta \, \hat{r} \ .$$

(c) Calculate the total radiated power in the same limit.

Answer:

$$P = \frac{cQ^2 s^4}{15} \left(\frac{\omega}{c}\right)^6 \ .$$